# THE CORONA CONJECTURE FOR A CLASS OF INFINITELY CONNECTED DOMAINS

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1. **Statement of results.** Let $D$ be a domain obtained from the open unit disk $\Delta$ by deleting a sequence of disjoint closed disks $\Delta_n$ converging to 0. We assume that the centers $c_n$ and radii $r_n$ of the $\Delta_n$ satisfy the following two conditions:

$$\text{(i)} \qquad \frac{|c_{n+1}|}{|c_n|} \leqq a < 1 \qquad \text{for all } n \geqq 1, \quad \text{and}$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} \frac{r_n}{|c_n|} < \infty .$$

Let $H^\infty(D)$ be the uniform algebra of bounded analytic functions on $D$ and let $\mathfrak{M}(H^\infty(D))$ be the maximal ideal space of $H^\infty(D)$. The Gleason parts of $H^\infty(D)$ are the equivalence classes in $\mathfrak{M}(H^\infty(D))$ defined by the relation $\|\phi-\psi\|<2$, where $\|\cdot\|$ is the norm in the dual of $H^\infty(D)$.

With the above assumptions on $D$ we have the following results.

THEOREM 1. *$D$ is dense in the maximal ideal space of $H^\infty(D)$.*

THEOREM 2. *The Gleason parts of $H^\infty(D)$ are all one-point parts or analytic disks, with the exception of the part containing $D$.*

The set of homomorphisms $\phi$ of $H^\infty(D)$ for which $\phi(z)=0$, where $z$ is the coordinate function on $D$, is called the "fiber over 0," and is designated by $\mathfrak{M}_0$. $\mathfrak{M}_0$ contains the "distinguished homomorphism" $\phi_0$ defined by

$$\phi_0(f) = \frac{1}{2\pi i}\int_{bD} \frac{f(z)dz}{z} .$$

If $z$ tends to zero in such a way that

$$\lim_{N\to\infty}\left(\lim_{z\to 0}\ \inf_{n\geqq N} \frac{|z-c_n|}{r_n}\right) = \infty$$

then $f(z)$ tends to $\phi_0(f)$ for all $f\in H^\infty(D)$, that is, $z$ tends to $\phi_0$ in $\mathfrak{M}(H^\infty(D))$. $\phi_0$ is in the same Gleason part as $D$ (cf. [5]).

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Let $N$ be the set of nonnegative integers, $\beta N$ its Čech compactification, and $\tilde{\beta}N=\beta N\backslash N$.

THEOREM 3. *The Gleason part containing $D$ is the union of $D$ and a subset $E$ of the fiber over zero. The set $E$ is homeomorphic to the quotient space obtained from $\tilde{\beta}(N)\times\Delta$ by identifying $\tilde{\beta}(N)\times\{0\}$ to the point $\phi_0$. Each of the functions in $H^\infty(D)$ is analytic on each slice of $\tilde{\beta}(N)\times\Delta$.*

The remainder of this note will be devoted to indicating how these theorems are proved, and how they can be extended to more general domains.

2. **The algebra** $H^\infty(\Delta\times N)$. The algebra $H^\infty(\Delta\times N)$ of bounded functions on $\Delta\times N$ which are analytic on each slice $\Delta\times\{n\}$ becomes a Banach algebra, when endowed with the supremum norm.

LEMMA 1. *$\Delta\times N$ is dense in the maximal ideal space $\mathfrak{M}(H^\infty(\Delta\times N))$ of $H^\infty(\Delta\times N)$.*

PROOF. Suppose $f_1, \cdots, f_n\in H^\infty(\Delta\times N)$ satisfy $|f_1|+\cdots+|f_n|\geqq\delta>0$ on $\Delta\times N$. We must find $g_1,\cdots,g_n\in H^\infty(\Delta\times N)$ satisfying $\sum f_jg_j=1$. By Carleson's solution of the corona conjecture for the unit disc $\Delta$, there are functions $g_{1m},\cdots,g_{nm}\in H^\infty(\Delta\times\{m\})$, such that $\sum_{j=1}^n f_jg_{jm}=1$ on $\Delta\times\{m\}$, and such that $|g_{jm}|\leqq M$, where $M$ depends only on $\delta$. The $g_{jm}$ then determine functions $g_j\in H^\infty(\Delta\times N)$ which do the trick.

Now $H^\infty(\Delta)$ can be considered a subalgebra of $C(Y)$, where $Y$ is the maximal ideal space of $L^\infty(b\Delta, d\theta)$. In fact, $H^\infty(\Delta)$ is a strongly logmodular algebra on $Y$, in the sense that every $u\in C_R(Y)$ is equal to $\log|f|$, for some $f\in H^\infty(\Delta)$. Regarding $H^\infty(\Delta\times\{m\})$ as a subalgebra of $C(Y\times\{m\})$, we see that $H^\infty(\Delta\times N)$ becomes an algebra on the Čech compactification $\beta(Y\times N)$ of $Y\times N$.

LEMMA 2. *$H^\infty(\Delta\times N)$ is a strongly logmodular algebra on $\beta(Y\times N)$.*

PROOF. Let $u\in C_R(\beta(Y\times N))$, and let $u_m$ be the restriction of $u$ to $Y\times\{m\}$. There is $f_m\in H^\infty(\Delta\times\{m\})$ such that $\log|f_m|=u_m$, regarded as functions on $Y\times\{m\}$. The $f_m$ determine a function $f\in H^\infty(\Delta\times N)$ such that $\log|f|=u$ on $Y\times N$, and hence on $\beta(Y\times N)$. That does it.

Now consider the function $Z\in H^\infty(\Delta\times N)$, defined by $Z(\lambda, n)=\lambda$. Then $\|Z\|=1$. The Gelfand transform of $Z$ will be denoted by $\hat{Z}$.

LEMMA 3. *The subset of $\mathfrak{M}(H^\infty(\Delta\times N))$ on which $|\hat{Z}|<1$ is homeomorphic to $\Delta\times\beta(N)$.*

PROOF. To each pair $(\lambda, p)\in\Delta\times\beta(N)$ corresponds the homomorphism of $H^\infty(\Delta\times N)$ which assigns to $f=\{f_j\}_{j=1}^\infty\in H^\infty(\Delta\times N)$ the value of the bounded sequence $\{f_j(\lambda)\}_{j=1}^\infty$ at $p$. This correspondence is easily seen to be the desired homeomorphism.

Now the closure of each $\Delta\times\{m\}$ is an open subset of $\mathfrak{M}(H^\infty(\Delta\times N))$. Let $X$ be the space obtained from $\mathfrak{M}(H^\infty(\Delta\times N))$ by deleting the closures of the $\Delta\times\{m\}$, $m\geqq 1$, and by identifying $\hat{Z}^{-1}(0)$ to a point. Let $A$ be the subalgebra of $C(X)$ obtained by restricting to $X$ the functions in $H^\infty(\Delta\times N)$ which are constant on $\hat{Z}^{-1}(0)$. In other words, $A$ is the linear span of $ZH^\infty(\Delta\times N)$ and the constants, regarded as continuous functions on $X$.

LEMMA 4. *$A$ is a uniform algebra on $X$, whose maximal ideal space is $X$. The set $E\subseteq X$ on which $|\hat{Z}|<1$ is homeomorphic to $\Delta\times\tilde{\beta}(N)$, with $\{0\}\times\tilde{\beta}(N)$ identified to a point. It forms a Gleason part of $A$. The remaining Gleason parts of $A$ are either points or analytic disks.*

PROOF. This lemma is easy to verify. The statement concerning the Gleason parts follows from the logmodularity of $H^\infty(\Delta\times N)$, and the embedding theorem for analytic disks (cf. [**3**]).

Note that $E$ is not dense in $X$. In fact, the function $f\in H^\infty(\Delta\times N)$, defined by $f(z,n)=z^n$, vanishes identically on $E$, while $|f|=1$ on $\beta(Y\times N)$, and hence on the Shilov boundary $\tilde{\beta}(Y\times N)$ of $A$.

3. **The isomorphism of the fiber and the fringe.** The pairwise disjoint sequence of disks $D_n^c$ with centers $c_n$ and radii $((1-a)/2)c_n$ have the property that every $f\in H^\infty(\Delta_n^c)$ for which $\|f\|\leqq 1$ satisfies $|f|\leqq(2/(1-a))\ r_n/|c_n|$ in $D_n$. This, together with condition (ii), gives

LEMMA 5. *If $\epsilon>0$ and $M>0$ are given, then there exists an integer $Q$ such that: If $f_n\in H^\infty(\Delta_n^c)$, $f_n(\infty)=0$, and $\|f_n\|\leqq M$, then*

$$\sum\{|f_m(z)| : m\geqq Q\}<\epsilon \qquad \textit{if } z\notin \cup\{D_n: n\geqq Q\},$$

$$\sum\{|f_m(z)| : m\geqq Q,\ m\neq n\}<\epsilon \qquad \textit{if } z\in D_n.$$

In particular, $\sum f_n$ converges uniformly on compact subsets of $D$ to a function $f\in H^\infty(D)$.

For $f\in H^\infty(D)$ define

$$(P_nf)(z)=\frac{1}{2\pi i}\int_{\partial\Delta_n}\frac{f(\xi)d\xi}{\xi-z}, \qquad z\in\Delta_n^c.$$

Then $P_n$ is a projection of $H^\infty(D)$ onto the functions in $H^\infty(\Delta_n^c)$ which vanish at $\infty$. Moreover $P_jP_k=0$ if $j\neq k$ and $\sup_n\|P_n\|<\infty$. If $\{f_n\}$ is as in Lemma 5, then $P_n(\sum f_m)=f_n$. Applying Lemma 5 to $\{P_nf\}$, we have that $\sum P_nf$ converges uniformly on compact subsets of $D$ to $f$ and $f(z)\to\phi_0(f)$ as $z\to 0$, $z\in D\backslash\bigcup D_n$. Further,

LEMMA 6. *Given $\epsilon>0$, there is a $Q$ such that for all $f\in H^\infty(D)$ $|f(z)-(P_nf)(z)-\phi_0(f)|<\epsilon$ for $n\geqq Q$ and $z\in D_n\backslash\Delta_n$.*

Let $L_n(z)=r_n/(z-c_n)$, and define

$$\Psi(f)(z,n)=(P_nf)(\overset{-1}{L_n}(z))+\phi_0(f),\qquad f\in H^\infty(D).$$

Then $\Psi$ is a continuous linear isomorphism of $H^\infty(D)$ and those functions in $H^\infty(\Delta\times N)$ which are constant on $\hat{Z}^{-1}(0)$. Moreover, if $f\in H^\infty(D)$ vanishes on the fiber over 0, then $\Psi(f)$ vanishes on the "fringe"

$$\mathfrak{M}(H^\infty(\Delta\times N))\Big\backslash\left(\bigcup_{n=1}^{\infty}\overline{\Delta\times\{n\}}\right).$$

Hence $\Psi$ determines a continuous linear operator $\Theta$ from $H^\infty|\mathfrak{M}_0$ to the algebra $A$ defined in Lemma 4.

LEMMA 7. *The map $\Theta$ from $H^\infty(D)|\mathfrak{M}_0$ to $A$ is an isometric (algebra) isomorphism.*

PROOF. By Lemma 6, there is a $Q$ for which

$$|P_n(fg)+\phi_0(fg)-(P_nf+\phi_0(f))(P_ng+\phi_0(g))|<\epsilon$$

if $n\geqq Q$ and $z\in D_n\backslash\Delta_n$. Composing with $L_n^{-1}$ and using the maximum modulus principle in $\Delta$ we have that

$$|\Psi(fg)(z,n)-\Psi(f)(z,n)\Psi(g)(z,n)|<\epsilon$$

for large $n$, and hence that $\Theta$ is multiplicative. That $\Theta$ is isometric follows easily from Lemma 6, the fact that $f(z)\to\phi_0(f)$ as $z\to 0$, $z\in D\backslash\bigcup D_n$, and the fact [5] that $\|\hat{f}\|_{\mathfrak{M}_0}=\lim\sup_{D\ni z\to 0}|f(z)|$ for all $f\in H^\infty(D)$. As was noted after Lemma 5, $P_n(\sum f_m)=f_n$, so that the image of $\Psi$ covers $ZH^\infty(\Delta\times N)$ and hence $\Theta$ is onto $A$.

PROOF OF THEOREM 1. Let $\phi$ be a homomorphism of $H^\infty(D)$. If $\phi$ is not in the fiber at 0 then Carleson's corona theorem can be used to show that $\phi$ is in the closure of $D$. Also, $\phi_0$ is the closure of $D$. Assume that $\phi\neq\phi_0$ is in the fiber over 0. By Lemma 7, $\phi$ defines a homomorphism of $A$ and hence a homomorphism $\tilde{\phi}$ of $H^\infty(\Delta\times N)$. $\tilde{\phi}$ is characterized by the fact that $\tilde{\phi}(\Psi(f))=\phi(f)$ for all $f\in H^\infty(D)$. Recall that $Z\in H^\infty(\Delta\times N)$ is the function defined by $Z(\lambda,n)=\lambda$. For $p\in N$, define $I_p \in H^\infty(\Delta \times N)$ by setting $I_p(\lambda, n) = 1$ if $n \geqq p$, and $I_p(\lambda, n) = 0$ if $n < p$. $\tilde{\phi}$ will satisfy $\tilde{\phi}(Z) \neq 0$ and, for each $p \in N$, $\tilde{\phi}(I_p) = 1$.

Let $f_1, \cdots, f_k \in H^\infty(D)$ and $0 < \epsilon < |\tilde{\phi}(Z)|/2$ be given. We will show there is a point $z \in D$ for which $|f_i(z) - \phi(f_i)| < 2\epsilon$ for $i = 1, \cdots, k$. Let $p \in N$. By the density of $\Delta \times N$ in $\mathfrak{M}(H^\infty(\Delta \times N))$ there is a $(\lambda, n) \in \Delta \times N$ with $|\Psi(f_i)(\lambda, n) - \tilde{\phi}(\Psi(f_i))| < \epsilon$ for $1 \leqq i \leqq k$, $|Z(\lambda, n) - \tilde{\phi}(Z)| < \epsilon$ and, $|I_p(\lambda, n) - \tilde{\phi}(I_p)| < \epsilon$. In particular, $|\lambda| > |\tilde{\phi}(Z)|/2$, and $n \geqq p$. If $p$ was chosen sufficiently large, the last two inequalities guarantee that $L_n^{-1}(\lambda) \in D_n \backslash \Delta_n$. Hence, if $p$ is also larger than the $Q$ of Lemma 6, then $L_n^{-1}(\lambda) \in D$ and

$$\begin{aligned} &|f_i(L_n^{-1}(\lambda)) - \phi(f_i)| \\ &\quad = |f_i(L_n^{-1}(\lambda)) - (P_n f_i)(L_n^{-1}(\lambda)) - \phi_0(f_i) + \Psi(f_i)(\lambda, n) - \tilde{\phi}(\Psi(f_i))| \\ &\quad \leqq 2\epsilon \quad \text{for } i = 1, \cdots, k. \end{aligned}$$

4. **Results for more general domains.** These same techniques can be used to prove the following corona theorem.

THEOREM. *Let $E$ be a domain for which $E$ is dense in $\mathfrak{M}(H^\infty(E))$. Let $D$ be a domain obtained from $E$ by excising a sequence of disjoint closed disks $\Delta(c_n; r_n)$, which satisfy the following conditions:*

(i) *There exists a disjoint sequence of disks $\Delta(c_n; s_n)$ contained in $E$ with $\sum r_n/s_n < \infty$,*

(ii) *$bE$ contains all the limit points of $\{c_n\}$.*

*Then $D$ is dense in $\mathfrak{M}(H^\infty(D))$.*

Note that this theorem includes Theorem 1 by taking $E = \Delta \backslash \{0\}$. The proof involves describing the maximal ideal spaces of the fibers of $H^\infty(D)$ over $\partial D$. These fibers become immensely more complicated, though, in the general case, than in the simple case we have described.

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