NONEXPANSIVE RETRACTS OF BANACH SPACES

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In what follows, C is a closed convex subset of the real, reflexive, strictly convex Banach space X. If $F \subset C$, we shall call F a nonexpansive retract of C if either $F = \emptyset$ or there is a retraction of C onto F which is a nonexpansive mapping.

THEOREM 1. *If T: $C \to C$ is nonexpansive, then $F(T)$, the fixed point set of T, is a nonexpansive retract of C.*

THEOREM 2. *The class of nonexpansive retracts of C is closed under arbitrary intersections.*

To prove these theorems, suppose F is a nonempty subset of C, and set $\mathfrak{F} = \{f: C \to C \mid f \text{ is nonexpansive and } F \subset F(f)\}$. Define an order on $\mathfrak{F}$ by setting $f < g$ if $\|fx - fy\| \leqq \|gx - gy\|$ for all $(x, y) \in C \times C$, with strict inequality holding for at least one pair (x, y); then set $f \leqq g$ to mean $f < g$ or $f = g$. Then $\leqq$ is a partial ordering of $\mathfrak{F}$.

Every linearly ordered subset of $\mathfrak{F}$ has a lower bound in $\mathfrak{F}$; the proof of this fact utilizes the local weak compactness of C and the weak lower semicontinuity of the norm. Therefore, by Zorn's lemma, $\mathfrak{F}$ has a minimal element.

The strict convexity of X implies that for each $g \in \mathfrak{F}$ there exists a $g_0 \in \mathfrak{F}$ with $F(g_0) = F(g)$ and such that whenever $\|g_0(u) - g_0(w)\| = \|u - w\|$, then $g_0(u) - g_0(w) = u - w$. For example, we may take $g_0 = \frac{1}{2}I + \frac{1}{2}g$, where I is the identity function for C.

Suppose f is a minimal function in $\mathfrak{F}$, and g is any function of $\mathfrak{F}$. Let g_0 be the function of the preceding paragraph; then $g_0 f \in \mathfrak{F}$ while $g_0 f \leqq f$. By the minimality of f, therefore $g_0 f = f$.

Letting $R(f)$ denote the range of f, therefore

$$F(f) \subset R(f) \subset F(g_0) = F(g), \tag{1}$$

and in particular,

$$F(f) \subset F(g) \qquad \text{for } g \in \mathfrak{F}. \tag{2}$$

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Taking $g=f$ in (1), we see that $F(f)=R(f)$, so that f is a nonexpansive retraction onto $F(f)$. From (2), if f and g are minimal elements of $\mathfrak{F}$, then $F(f)=F(g)$.

We claim that this common set $F(f)$ is the smallest nonexpansive retract F' of C with $F'\supset F$. If the g of (2) is any nonexpansive retraction with $F(g)=F'$, we have from (2) that $F\subset F(f)\subset F'$, which is just our claim.

To prove Theorem 1, suppose $F(T)\neq\emptyset$. Set $F=F(T)$ and let f be a minimal element of $\mathfrak{F}$. Taking $g=T$ in (2), $F(f)\subset F(T)$, while $F(T)\subset F(f)$ in order for $f\in\mathfrak{F}$; thus f is a nonexpansive retraction of C onto $F(f)=F(T)$. q.e.d.

To prove Theorem 2, suppose F_λ is a nonexpansive retract of C for $\lambda\in\Lambda$. Set $F=\bigcap_\lambda F_\lambda$; we may suppose $F\neq\emptyset$. We have already remarked that if f is a minimal element of $\mathfrak{F}$, then $F\subset F(f)\subset F'$ for all nonexpansive retracts F'; in particular, $F(f)\subset F_\lambda$ for each λ, so

$$F \subset F(f) \subset \bigcap_\lambda F_\lambda = F,$$

and f is the required nonexpansive retraction. q.e.d.

A retraction f of C onto F will be called a *ray retraction* if whenever $q\in C$ is on the ray from $f(p)$ through p, we have $f(q)=f(p)$.

THEOREM 3. *Suppose X^* is strictly convex and F is a nonempty nonexpansive retract of C. Then there is at most one nonexpansive ray retraction f of C onto F; if it exists it must satisfy*

$$\|f(p)-f(q)\|^2 \leqq (J(f(p)-f(q)),\, p-q) \tag{3}$$

for all p, q in C. Conversely, a retraction satisfying (3) *is a nonexpansive ray retraction.*

A nonexpansive ray retraction is known to exist if:

(a) *$F\cap B$ is strongly compact for each ball B in X, or*

(b) *X is uniformly convex and $Jx_n\rightharpoonup 0$ in X^* whenever $x_n\rightharpoonup 0$ in X.*

(Here $J: X\to X^*$ is the normalized duality mapping and $\rightharpoonup$ denotes weak convergence.)

The proof of Theorem 3 is substantially different from the proofs of the other theorems; it utilizes an approximation scheme of F. E. Browder [1] to construct nonexpansive mappings x_λ: $C\to C$ satisfying

$$x_\lambda(p) = \lambda\cdot g(x_\lambda(p)) + (1-\lambda)\cdot p$$

for $0<\lambda<1$, $p\in C$, where g is a nonexpansive retraction of C onto F. It is then shown that under hypothesis (a) or (b), a strong $\lim_{\lambda\to 1} x_\lambda(p)$ $=f(p)$ exists and when such a strong limit exists, f satisfies condition

(3) of the theorem. Furthermore, if g is already a nonexpansive ray retraction, then $s-\lim_{\lambda\to 1} x_\lambda(p)=g(p)$; thus nonexpansive ray retractions must satisfy (3).

It is expected that detailed proofs of these and related theorems will appear elsewhere.

Reference

1. F. E. Browder, *Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces*, Arch. Rational Mech. Anal. 24 (1967), 82–90. MR **34** #6582.

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