EXTREME POINTS OF THE SET OF UNIVALENT FUNCTIONS

BY LOUIS BRICKMAN[1]

Communicated by Irving Glicksberg, August 18, 1969

Let S be the usual set of analytic, normalized, univalent functions on the unit disk. A function belonging to S is called an *extreme point of* S if it cannot be written as a proper convex combination of two other members of S. (See [**1**, p. 439].) For example the Koebe function is an extreme point of S, a fact that follows, for instance, from the unique maximal property of the second coefficient. We shall let E denote the set of extreme points of S.

The set of all analytic functions on the unit disk is a locally convex linear topological space, and S is a compact subset [**2**, p. 217]. Therefore the conclusion of the Krein-Milman theorem [**1**, p. 440] applies. Namely, $S\subset \mathrm{Cl}(\mathrm{co}\, E)$. In other words every function in S is the limit, uniform on each compact subset of the disk, of a suitable sequence of convex combinations of extreme points. Thus, the determination of of E should provide a tremendous amount of information about S. For example, the Bieberbach conjecture for the functions of E implies the full conjecture. In fact any continuous linear functional achieves its maximum real part, maximum modulus, etc. on S at a point of E. These statements follow from Lemma 2 [**1**, p. 439] and Lemma 3 [**1**, p. 440]. Although we have been unable to characterize the functions belonging to E, the theorem below provides a very simple necessary condition on the range of such a function. It is believed that the techniques used here can be applied further to refine this result.

We wish to thank Professors Thomas H. MacGregor and Donald R. Wilken for many stimulating conversations. In particular Wilken first raised the question of determining the extreme points of S, and MacGregor showed that severe restrictions on the range of an extreme point could be expected.

LEMMA 1. *Let $f\in S$. Suppose there is a function ϕ, analytic on range f but not of the form $\phi(w)=aw+b$, and two complex numbers, α and β, such that*

$$(\phi(w_1)-\phi(w_2)/(w_1-w_2)\neq\alpha,\quad (\phi(w_1)-\phi(w_2))/(w_1-w_2)\neq\beta$$

AMS Subject Classifications. Primary 3042.

Key Words and Phrases. Extreme points, convex combinations, normalized univalent functions.

[1] Supported by the National Science Foundation grant GP 12017.

for all w_1 and w_2 in range f, $w_1 \neq w_2$. Suppose further that $\phi'(0) = t\alpha + (1-t)\beta$ for some t satisfying $0 < t < 1$. Then $f \notin E$.

PROOF. The functions $\alpha w - \phi(w)$ and $\beta w - \phi(w)$ are univalent on range f. Therefore so are the normalized functions

$$\phi_1(w) = \frac{\alpha w - \phi(w) + \phi(0)}{\alpha - \phi'(0)}, \qquad \phi_2(w) = \frac{\beta w - \phi(w) + \phi(0)}{\beta - \phi'(0)}.$$

Also,

$$\{\alpha - \phi'(0)\}\phi_1(w) + \{\phi'(0) - \beta\}\phi_2(w) = (\alpha - \beta)w.$$

Therefore

$$w = (1 - t)\phi_1(w) + t\phi_2(w),$$
$$f(z) = (1 - t)\phi_1(f(z)) + t\phi_2(f(z)).$$

This exhibits f as a convex combination of two functions of S, and the assumption $\phi(w) \neq aw + b$ insures that the two functions are different. Thus, $f \notin E$.

LEMMA 2. *Let $f \in S$. If there are two numbers of equal modulus not in range f, then $f \notin E$.*

PROOF. Let $a = re^{i\alpha}$ and $b = re^{i\beta}$ be two points not in range f, where $r > 0$ and $0 < \alpha - \beta < 2\pi$. Since range f is a simply connected region, $\exists\phi$, analytic on range f, such that $\{\phi(w)\}^2 = (w-a)(w-b)$. Then $\phi'(0) = -(a+b)/2\phi(0) = \pm\cos\frac{1}{2}(\alpha - \beta)$. Therefore $-1 < \phi'(0) < 1$.

But a straightforward calculation, using only the relation $a \neq b$, shows that

$$(\phi(w_1) - \phi(w_2)/(w_1 - w_2) \neq \pm 1 \qquad (w_1 \neq w_2).$$

Hence the desired conclusion follows from Lemma 1.

THEOREM. *Let $f \in E$. Then f maps the unit disk onto the complement of a continuous curve tending to infinity with increasing modulus.*

PROOF. Let Γ denote the set of complex numbers not in range f. Let $r_0 = \inf\{|w| : w \in \Gamma\}$. Since range f is a simply connected region there exists $w \in \Gamma$ such that $|w| = r$ for every $r \geqq r_0$. By Lemma 2 there is at most one such w. Hence the absolute value function maps Γ onto $[r_0, \infty)$ in one-to-one fashion. Let γ be the inverse function. For every $r \geqq r_0$, the set $\{w \in \Gamma : r_0 \leqq |w| \leqq r\}$ is compact. Hence the restriction of γ to $[r_0, r]$ is the inverse of a continuous function with compact domain. It follows that γ is continuous on $[r_0, r]$ and, therefore, on $[r_0, \infty)$ as required.

Regarding the coefficient problem, it is interesting to compare this theorem with the known result [3] that any function in S which maximizes $|a_n|$ (for any fixed n) must map the disk onto the complement of an analytic arc ending with an asymptote at infinity. Hence there exists a function that maximizes $|a_n|$ having both this property and the property of the above theorem.

BIBLIOGRAPHY

1. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

2. Z. Nahari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR **13**, 640.

3. M. Schiffer, *On the coefficient problem for univalent functions*, Trans. Amer. Math. Soc. **134** (1968), 95–101. MR **37** #4249.

STATE UNIVERSITY OF NEW YORK, ALBANY, NEW YORK 12203