CONTINUOUS SELECTION OF REPRESENTING MEASURES[1]

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Let B be a linear subspace of $C_R(X)$, the continuous real functions on a compact space X, and let Γ be the Šilov boundary of B in X. We give here conditions which are sufficient for there to be an integral representation of the form

$$(1) \qquad u(x) = \int_\Gamma u(\theta) g_x(\theta) d\mu(\theta),$$

where $x \rightarrow g_x$ is a continuous map from some subset Δ of X into $L_\infty(\mu)$. With the additional condition that Δ is separable, we obtain a kernel representation of the form

$$(2) \qquad u(x) = \int_\Gamma u(\theta) Q(x, \theta) d\mu(\theta)$$

where Q is a continuous function of x and $x \rightarrow Q(x, \cdot)$ is continuous with respect to the $L_\infty(\mu)$ norm. If it is also the case that $B|\Gamma$ is dense in $L_1(\mu)$, then $Q(\cdot, \theta)$ is a limit (uniform convergence on compact subsets of Δ) of functions in B. These results also give integral representations like (1) and (2) for a complex function algebra, simply by considering the space B of real parts of the algebra. The details of this work will appear in [3].

We use the following notation throughout this paper:

X is a compact Hausdorff space, with topology $\mathfrak{I}$.

B is a linear subspace of $C_R(X)$, containing the constant functions, and separating the points of X.

Γ is the Šilov boundary of B in X.

$\overline{B}_\Delta$, for any set $\Delta \subset X$, is the closure of $B|\Delta$ in the topology of uniform convergence on compact subsets of Δ.

$B^+(\Delta, z) = \{u|\Delta : u \in B,\ u > 0,\ u(z) = 1\}$.

($\overline{B}_\Delta$ is an abstract version of "all harmonic functions on Δ," and $B^+(\Delta, z)$ is the set of normalized-at-z positive B-functions.)

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A *representing measure* ν for $x \in X$ is a positive Baire probability measure on Γ such that $u(x) = \int_\Gamma u d\nu$ for all $u \in B$.

P_x is the set of all representing measures for x.

$B^\perp$ is the set of real signed measures on Γ which are orthogonal to B.

B' is the space of sup-norm continuous linear functionals on B, with the w^*-topology = the $w(B', B)$ topology = the topology of pointwise convergence on B.

$T_B = \{F \in B' : F(1) = \|F\| = 1\}$. T_B is a compact convex set in B'. We consider X embedded homeomorphically in T_B. Then Γ is the closure of the extreme points of T_B, and B is isometric to the restriction to T_B or X or Γ of all w^*-continuous linear functionals on B'.

We use x, y, z etc. for points of T_B, and write $u(x)$ for $x(u)$, where $x \in T_B$ and $u \in B$.

Write $x \sim y$ for $x, y \in T_B$ iff there is a number a such that

$$a^{-1} < u(x)/u(y) < a \tag{3}$$

for all strictly positive $u \in B$. The relation $\sim$ is an equivalence relation, and the equivalence classes are called the Gleason parts of X or T_B. The parts of X are the intersections of X with the parts of T_B. The Gleason parts of a complex function algebra A are the same as the parts defined by the space $B = \text{Re } A$.

On each part Δ of X or T_B define a metric as follows:

$d(x, y) = \sup\{|\log u(x) - \log u(y)| : u \in B, u > 0\}$. Write $d(x, y) = \infty$ if $x \nsim y$. Let $\mathfrak{J}_d$ be the d-metric topology on Δ.

The parts of T_B are convex subsets which can be characterized as follows: $x \sim y$ iff the segment $[x, y]$ extends some distance beyond x and y in T_B; i.e., if

$$\begin{aligned} (1 + r)x - ry &\in T_B, \\ (1 + r)y - rx &\in T_B, \end{aligned} \tag{4}$$

for some $r > 0$. Then we also have

$$d(x, y) = \inf\{\log(1 + 1/r) : r \text{ satisfies (4)}\}. \tag{5}$$

We use this definition of part and part metric d in any convex set containing no whole line.

Let $C = C_R(\Gamma)$, so that T_C can be identified with the positive probability measures on Γ. The part Π_μ of T_C containing μ is the set of all measures $g\mu$, where g is a positive function of $L_\infty(\mu)$ which is bounded away from zero.

We use D for the part metric in parts of T_C. Convergence in D is equivalent to convergence of the corresponding g's in $L_\infty(\mu)$. We write $D(g, h)$ instead of $D(g\mu, h\mu)$ for points $g\mu, h\mu$ of T_C.

The part metric on any part of a function space is complete. That is d is complete on any part Δ of X or T_B, and D is complete on parts Π_μ of T_C.

Let Δ be a part of X with part metric d. Then $\mathfrak{Z}=\mathfrak{Z}_d$ iff $B^+(\Delta, z)$ is equicontinuous for some (every) $z\in\Delta$.

The following theorem indicates the extent to which our sufficient conditions are necessary.

THEOREM 1. *If Δ is a subset of X, and there is a continuous map $x\to g_x\mu$ on Δ to a part Π_μ of T_C (so that $x\to g_x$ is a continuous map into $L_\infty(\mu)$) and*

$$u(x) = \int_\Gamma u g_x d\mu \tag{6}$$

for all $u\in B$, $x\in\Delta$, then Δ is contained in a part of X, and $\mathfrak{Z}=\mathfrak{Z}_d$ on Δ.

For any convex set K, let K^i be the set of all $x\in K$ such that for every $y\in K$, the segment $[y, x]$ extends beyond x in K. If $K^i\neq 0$, then K^i is a part, which we call the *inner part* of K. (Incidentally, K is a convex body in some linear topological space iff $K^i\neq 0$.) Finite dimensional convex sets have nonempty inner parts.

THEOREM 2. *Let Δ be a part of X, with $\mathfrak{Z}=\mathfrak{Z}_d$ on Δ. Let $P_x^i\neq 0$ for some $x\in\Delta$. Then there is a continuous map $x\to g_x\mu$ on Δ into some part Π_μ of T_C such that $g_x\mu\in P_x$ for all $x\in\Delta$.*

Note. The condition $\mathfrak{Z}=\mathfrak{Z}_d$ can be replaced by the assumption that $B^+(\Delta, z)$ is equicontinuous. The condition $P_x^i\neq 0$ is implied by the assumption that $B^\perp$ is finite dimensional. The measure $g_x\mu$ is a D-continuous function of x, which implies that g_x is an L_∞-continuous function of x.

INDICATION OF PROOF. If $B^\perp$ is finite dimensional, then each P_x is, and hence $P_x^i\neq 0$. If $P_x^i\neq 0$ for some x, then $P_y^i\neq 0$ for all $y\sim x$, and the sets P_x^i, $x\in\Delta$, are all contained in one part Π_μ of T_C. This result is due to Har'kova [6], who makes ingeneous use of Bishop's theorem [5]. If $\hat{\Delta}$ is the part of T_B containing Δ, then $P_x^i\subset\Pi_\mu$ for all $x\in\hat{\Delta}$. The projection of Π_μ into T_B (restriction of the measures from $C=C_R(\Gamma)$ to B) is a D-d continuous affine map on Π_μ onto $\hat{\Delta}$. This restriction map is therefore open, since Π_μ, D and $\hat{\Delta}$, d are one-part complete convex sets [1]. The inverse map, $x\to P_x\cap\Pi_\mu=P_x^i$, is a lower semicontinuous map of $\hat{\Delta}$ into D-closed convex subsets of Π_μ. By an adaptation of Michael's continuous selection theorem to part-metric setting [2, Theorem 12] there is a continuous selection $x\to g_x\mu\in P_x^i$. Continuity with respect to d on Δ and D in Π_μ is the same as continuity with respect to $\mathfrak{J}$ on Δ, and the L_∞ norm applied to the functions g_x.

THEOREM 3. *Assume the hypotheses of Theorem 2, and in addition that Δ has a countable dense set. Then there is a measure μ on Γ and a jointly measurable function $Q(x, \theta)$ on $\Delta\times\Gamma$ such that $Q(x, \cdot)d\mu(\cdot)$ represents x, $x\to Q(x, \cdot)$ is L_∞-continuous, and $Q(\cdot, \theta)$ is continuous on Δ for each $\theta\in\Gamma$. If $B|\Gamma$ is dense in $L_1(\mu)$, then $Q(\cdot, \theta)\in\overline{B}_\Delta$ for each $\theta\in\Gamma$.*

PROOF. The proof of [4] can be modified to get from the measureable functions $g_x(\theta)$ to a kernel $Q(x, \theta)$ in the separable case. The fact that $Q(\cdot, \theta)\in\overline{B}_\Delta$ is also proved in [4], with an assumption that implies $B|\Gamma$ is dense in $L_1(\mu)$.

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