BOUNDING IMMERSIONS OF CODIMENSION 1 IN THE EUCLIDEAN SPACE

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Let M be an $(n+1)$-dimensional differentiable manifold without boundary (compact or not) and $f: V \to M$ an immersion of the compact n-dimensional manifold without boundary V. We say that f is a *bounding immersion* if there is a manifold W^{n+1} with boundary $dW = V$, and an immersion $g: W \to M$ such that $f = g|V$. If M and V are oriented, then V must be the oriented boundary of the oriented manifold W, and g an oriented immersion of codimension 0.

Using the classification of immersions (Smale [7], Hirsch [2]) and the work of Kervaire-Milnor [3], [4], we compute in this note the regular homotopy classes of all bounding immersions of the sphere S^n into the euclidean space R^{n+1} and into the sphere S^{n+1}.

1. **Statement of the results.** From [2] we know that the derivation $f \mapsto T(f)$ defines a weak homotopy equivalence between the space $\mathrm{Imm}(V, M)$ of the immersions of V into M and the space of the fibre-maps of the tangent bundle $T(V)$ into the tangent bundle $T(M)$ which are injective in each fibre. If $V = S^n$ and $M = R^{n+1}$, the set of connected components of this last space is an homogeneous space under the group $\pi_n(\mathrm{SO}(n+1))$. By a convenient identification, we obtain a bijection $\gamma: \pi_0(\mathrm{Imm}(S^n, R^{n+1})) \to \pi_n(\mathrm{SO}(n+1))$ such that the class of the ordinary imbedding be $0 \in \pi_n(\mathrm{SO}(n+1))$. Furthermore the map γ is additive with respect to the connected sum of immersions [5].

Similarly, using the fact that the fibration $\mathrm{SO}(n+2) \to S^{n+1} = \mathrm{SO}(n+2)/\mathrm{SO}(n+1)$ is the principal fibration with group $\mathrm{SO}(n+1)$ tangent to S^{n+1}, it is easy to obtain a bijection $\beta: \pi_0(\mathrm{Imm}(S^n, S^{n+1})) \to \pi_n(\mathrm{SO}(n+2))$ additive with respect to the connected sum. If $i: R^{n+1} \to S^{n+1}$ is the stereographic projection with the south pole $(x_1 = -1)$ as center, we have a commutative diagram

$$\begin{array}{ccc} \pi_0(\mathrm{Imm}(S^n, R^{n+1})) & \xrightarrow{\gamma} & \pi_n(\mathrm{SO}(n+1)) \\ \downarrow i_* & & \downarrow s \\ \pi_0(\mathrm{Imm}(S^n, S^{n+1})) & \xrightarrow{\beta} & \pi_n(\mathrm{SO}(n+2)) \end{array}$$

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where the stabilization homomorphism s is induced by the inclusion of $SO(n+1)$ as the subgroup of $SO(n+2)$ acting on the $n+1$ last coordinates. From now on, we shall denote by $\tilde{f}$ the class $\gamma(f)$.

If we denote by $J_n:\pi_n(SO(n+2))\to\pi_n$ $(=\pi_{2n+2}(S^{n+2}))$ the stable Hopf-Whitehead homomorphism, we can state the result:

THEOREM 1. *For each $n\geqq 1$, the set of the classes of the bounding immersions of S^n in S^{n+1} is the kernel of J_n.*

THEOREM 2. *For each $n\geqq 2$, the set of the classes of the bounding immersions of S^n in R^{n+1} is the kernel of $J_n\circ s$.*

In order to prove those results, we admit some lemmas whose proof will appear elsewhere.

2. **First step of the proof.** Let A_{n+1} be the cobordism group of stably parallelized manifolds W^{n+1} with boundary S^n. If W' is the manifold without boundary obtained from W by gluing a disk D^{n+1} along the boundary $S^n=dW$, we denote by $a(W, T)\in\pi_n(SO(n+2))$ the obstruction to extend the s-parallelization T of W to W': thence we have an homomorphism $a:A_{n+1}\to\pi_n(SO(n+1))$. Similarly, let B_{n+1} be the *monoïd* of isomorphism classes of such manifolds W with a true parallelization. It follows from [2] or [6] that, if t is a parallelization of W, there is an immersion $g: W\to R^{n+1}$, unique up to regular homotopy, such that the trivialization $T(g)$ of $T(W)$ be homotopic to t. If we consider the class of the restriction f of g to $dW=S^n$, we define an homomorphism $b:B_{n+1}\to\pi_n(SO(n+1))$. Furthermore we have a natural homomorphism $S:B_{n+1}\to A_{n+1}$.

LEMMA 1. *The following diagram*

$$\begin{array}{ccc} B_{n+1} & \xrightarrow{b} & \pi_n(SO(n+1)) \\ S\downarrow & & s\downarrow \\ A_{n+1} & \xrightarrow{a} & \pi_n(SO(n+2)) \end{array}$$

is commutative.

Thus, the set of classes of bounding immersions in $\mathrm{Imm}(S^n, R^{n+1})$, which is the image of b, is a monoïd included in $\mathrm{Ker}(J_n\circ s)$ because of the exactness of the sequence

$$A_{n+1}\xrightarrow{a}\pi_n(SO(n+2))\xrightarrow{J_n}\pi_n$$

(see [4]); and $b(B_{n+1})$ intersects each fibre $s^{-1}(x)$, $x\in\mathrm{Ker}(J_n)$, since the map S is surjective. To prove Theorem 2, it suffices to prove that $b(B_{n+1})$ contains $\mathrm{Ker}(s)$ (if $n\geqq 2$).

3. **Second step.** Let $u \in \pi_n(\mathrm{SO}(n+1))$ be the boundary of the generator $i_{n+1} \in \pi_{n+1}(S^{n+1})$ in the homotopy exact sequence

$$\pi_{n+1}(S^{n+1}) \xrightarrow{d} \pi_n(\mathrm{SO}(n+1)) \xrightarrow{s} \pi_n(\mathrm{SO}(n+2)) \to 0$$

of the fibration $S^{n+1} = \mathrm{SO}(n+2)/\mathrm{SO}(n+1)$. The cyclic group $\mathrm{Ker}(s)$ is generated by u. From the following lemma and the fact that there are parallelizable closed manifolds in all dimensions, it results that u is the class of a bounding immersion:

LEMMA 2. *If W'^{n+1} is a closed parallelizable closed manifold, and t is the restriction to $W = W' - D^{n+1}$ of a parallelization of W', then $b(W, t) = u \in \pi_n(\mathrm{SO}(n+1))$.*

Now, we can prove Theorem 1. First, we remark that any immersion $F: S^n \to S^{n+1}$ is regular homotopic to an immersion $i \circ f$, where $f \in \mathrm{Imm}(S^n, R^{n+1})$ and that $i \circ f$ and $i \circ f'$ have the same class in $\mathrm{Imm}(S^n, S^{n+1})$ if and only if there is some $q \in Z$ such that $\tilde{f}' = \tilde{f} + qu$ $(\in \pi_n(\mathrm{SO}(n+1)))$. Then we remark that, if F is a bounding immersion in S^{n+1}, it is regular homotopic to an immersion F' bounded by $G': W \to S^{n+1}$ whose image $G'(W)$ avoids the south pole. Therefore:

LEMMA 3. *Let $f \in \mathrm{Imm}(S^n, R^{n+1})$; the following assertions are equivalent:*

(i) $J_n \circ s(\tilde{f}) = 0$.

(ii) *There is a bounding immersion regular homotopic (in S^{n+1}) to $i \circ f$.*

(iii) *There is a bounding immersion $f' \in \mathrm{Imm}(S^n, R^{n+1})$ such that $\tilde{f}' = \tilde{f} + qu$ for some $q \in Z$.*

Theorem 1 is a quite evident consequence of Lemma 3.

4. **Last step.** *If n is even,* Theorem 2 is already proved, because $\mathrm{Ker}(s)$ contains at most the two elements 0 and u which are both bounding. If $n = 2$ or 6, then $\pi_n(\mathrm{SO}(n+1)) = 0$ and the only class is trivially the class of a bounding immersion. If $n \neq 2, 6$, then J_n is injective [1] and the two distinct classes 0 and u are the only bounding classes.

If n is odd, the kernel of s is infinite cyclic, generated by u and it suffices to prove that $-u$ is the class of a bounding immersion, since $b(B_{n+1})$ is a monoïd.

If $f \in \mathrm{Imm}(S^n, R^{n+1})$, let $d(\tilde{f}) \in Z$ be the normal degree (curvatura integra) of the immersion f (see [5]). It is proved in [5] that $d(\tilde{f} + \tilde{f}') = d(\tilde{f}) + d(\tilde{f}') - 1$. Now, the Hopf theorem of curvatura integra states that $d(\tilde{f}) = \chi(W)$ if f is the restriction to the boundary of an immersion

$g: W \to R^{n+1}$. It is clear that $d(0) = 1$, and it follows from Lemma 2 that $d(u) = -1$. Thus, the elements qu ($q \in Z$) of Ker(s) are determined by their (odd) degree $d(q \cdot u) = 1 - 2q$.

If $n = 1$, there is no 2-manifold, with boundary S^1, whose Euler number is more than 1, so that:

THEOREM 2'. *In* $\pi_0(\mathrm{Imm}(S^1, R^2)) \cong \pi_1(\mathrm{SO}(2))$, *the classes of bounding immersions are the classes of odd degree* $1 - 2q$, $q \geqq 0$.

For n odd $\neq 1$, the manifold $W' = S^2 \times S^{n-1}$ is s-parallelizable; there is a parallelization t of the manifold $W = W' - D^{n+1}$ which stably extend to W'. It follows from Lemma 1 that $b(W, t) \in \mathrm{Ker}(s)$. Now, the Euler number of W is 3 so that $b(W, t) = -u$. Thus, $-u$ is the class of a bounding immersion and Theorem 2 is proved.

5. Application.

THEOREM 3. *Let* V^n *be an s-parallelizable compact manifold without boundary, and* $f: V \to R^{n+1}$ *an immersion. Suppose* $n \geqq 2$. *If* $i \circ f: V \to S^{n+1}$ *is a bounding immersion, then f is regular homotopic (in* R^{n+1}*) to an immersion* f' *which is bounding (in* R^{n+1}*).*

If the manifold V is the n-sphere, this theorem is an immediate corollary of Theorems 1 and 2. In the general case, we deform the immersion $G: W \to S^{n+1}$ which bounds $F = i \circ f$ in an immersion G' whose image $G'(W)$ avoid the south pole, so that $G' = i \circ g'$. The immersion g' bounds f' such that $i \circ f' = F' = G' | V$. But f and f' have not the same class (in R^{n+1}) because, during the regular homotopy, the class of f has been changed by each crossing of the south pole.

Let $F_t: V \to S^{n+1}$ ($t \in [0, 1]$) be a regular homotopy with only one crossing of the south pole through $F_t(V)$, then f_1 is regular homotopic to the connected sum $f_0 + h$ of f_0 with an immersion $h: S^n \to R^{n+1}$ with class $\tilde{h} \in \mathrm{Ker}(s)$ (in fact, $\tilde{h} = \pm u$, depending on the direction of the crossing).

Thus, f is regular homotopic to an immersion f'' which is the connected sum of f' with some immersions h_i such that $\tilde{h}_i \in \mathrm{Ker}(s)$. We can replace the h_i by bounding immersions k_i of the same class (Theorem 2), and, now, f'' is the connected sum of the bounding immersions f' and k_i; so f'' is a bounding immersion.

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