A CONJECTURE CONCERNING TRANSITIVE SUB-ALGEBRAS OF LIE ALGEBRAS

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This is to announce the settling of the following conjecture: Given a Lie pseudogroup [1] acting transitely on a manifold, is there a finite-dimensional subgroup which also acts transitively? The answer is, in general, no. We give here an example and, in addition, give the Jordan-Hölder decomposition of a large class of counter-examples. Finally, we show how these counterexamples occur among general transitive pseudogroups. Following [1] and [2], we work in the category of transitive (filtered) Lie algebras. Details will appear in a forthcoming paper [3].

A transitive algebra L is called *minimal* if, given a transitive subspace T [1], L is the smallest transitive subalgebra generated by T.

THEOREM 1. *Every minimal ideal* [2] *of a minimal transitive Lie algebra is abelian.*

According to the results of [2], this theorem is proved if it can be shown that a minimal ideal cannot be (a) a simple transitive Lie algebra or (b) a simple intransitive Lie algebra. This is accomplished for (a) by using the results of [4] and for (b) by applying the spectral sequence for ideals in Lie algebras [5] together with some of the techniques of [4]. The classification of the simple infinite-dimensional Lie algebras [6] is used repeatedly.

Using Theorem 1 it is not hard to prove

THEOREM 2. *Every minimal transitive Lie algebra L has the following Jordan-Hölder decomposition*:

$$L \supset I_1 \supset I_2 \supset I_3 \supset \cdots I_s \supset I_{s+1} = \{0\},$$

where I_n/I_{n+1} *is abelian and* L/I_1 *is either a simple Lie algebra or a finite-dimensional abelian Lie algebra.*

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As a result of Theorems 1 and 2, the simplest way to try to produce a minimal Lie algebra would be to take a simple Lie algebra, say sl(2, C), the Lie algebra of the special linear group in two variables, and an sl(2, C)-module I which is simple (that is, has no submodules except $\{0\}$ and I) and look at the semidirect product $L = \mathrm{sl}(2, C) \circledS I$. In most cases, L will not be minimal, because one will be able to find a fundamental subalgebra A such that I has many nontrivial submodules M with respect to the part H of sl(2, C) not in the fundamental subalgebra. $H \circledS M$ will then be a proper transitive subalgebra. In order to prevent this, one must have the two elements in sl(2, C) not in the Cartan subalgebra operating in such a way that neither of them preserves an open subspace of I. Upon being given these requirements, Thomas Sherman produced such a representation of sl(2, C): Let $I = \prod_{n=-\infty}^{\infty} Ce_n$, with the filtration on I defined as $I^i = \prod_{|n|>i} Ce_n$. Using E_+, E_-, H to denote a basis for sl(2, C), with $\{H\}$ the Cartan subalgebra, let

$$\begin{aligned} E_+e_n &= e_{n+1}, \\ E_-e_n &= (n^2 + \gamma)e_{n-1}, \quad \gamma \text{ an irrational number}, \\ He_n &= (2n + 1)e_n. \end{aligned}$$

A straightforward argument shows that this algebra is minimal.

Finally, the following information is obtained about the transitive Lie algebras which have no finite-dimensional transitive subalgebra. The *dimension* of a transitive Lie algebra is dimension L/L_0.

THEOREM 3. *Every transitive algebra of dimension two or one has a finite-dimensional transitive subalgebra.*

The proof makes use of Quillen's generalization of the Schur Lemma [7].

THEOREM 4. *Every transitive Lie algebra has a transitive subalgebra L with the following Jordan-Hölder decomposition*:

$$L \supset I_1 \supset I_2 \supset \cdots \supset I_n = \{0\},$$

where I_k/I_{k+1} is abelian, and L/I_1 is either

(a) *finite-dimensional abelian,*

(b) *finite-dimensional semisimple, or*

(c) *infinite-dimensional simple.*

REFERENCES

1. I. M. Singer and S. Sternberg, *The infinite groups of Lie and Cartan.* I: *The Transitive groups,* Analyse Math. 15 (1965), 1–114. MR 36 #911.

2. V. W. Guillemin, *A Jordan-Hölder decomposition for a certain class of infinite-dimensional Lie algebras*, J. Differential Geometry **2** (1968), 313–345.

3. C. Freifeld, *On transitive subalgebras of Lie algebras* (to appear).

4. ———, *The cohomology of transitive filtered modules*. I: *The first cohomology group*, Trans. Amer. Math. Soc. **144** (1969), 475–491.

5. G. Hochschild and J.-P. Serre, *Cohomology of Lie algebras*, Ann. of Math. (2) **57** (1953), 591–603. MR **14**, 943.

6. V. Guillemin, D. Quillen, and S. Sternberg, *The classification of the complex primitive infinite pseudogroups*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966), 687–690. MR **34** #3594.

7. D. Quillen, *On the endomorphism ring of a simple module over an enveloping algebra* (to appear).

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