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SIMILARITY OF CANONICAL MODELS

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1. **Introduction.** In the last decade, Sz.-Nagy and Foiaş [4], [7] and de Branges and Rovnyak [1], [2] have developed general structure theories for contraction operators on Hilbert space based on the notion of a canonical model. In each of these theories the canonical model for a completely nonunitary contraction T is a representation of T as a formally simple operator acting on a possibly complicated space. Associated with each canonical model is an operator valued analytic function called the characteristic operator function of the model. One of the general problems of model theory is to discover how properties of T are reflected in the characteristic operator function of its model. Our main result (Theorem 2) is a solution (in this sense) of the problem of similarity of two canonical models in the special case when the associated characteristic operator functions are *complex valued*. Our main tool is the Sz.-Nagy and Foiaş lifting theorem for intertwining maps (see [3], [8]).

2. **The main results.** Suppose that N_1 and N_2 are Hilbert spaces. $\mathcal{L}(N_1, N_2)$ denotes the Banach space of (bounded linear) operators from N_1 to N_2. If $T_i \in \mathcal{L}(N_i, N_i)$ $(i=1, 2)$, we denote by $\mathcal{I}(T_1, T_2)$ the subspace of intertwining maps from T_1 to T_2, i.e., $\mathcal{I}(T_1, T_2) = \{X \in \mathcal{L}(N_1, N_2): XT_1 = T_2X\}$.

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L^p $(1 \leqq p \leqq \infty)$ is the classical Lebesgue space of the unit circle C in the complex plane and H^p is its Hardy subspace (see [5]). The L^p norm of f is denoted by $\|f\|_p$. If E is a measurable subset of C, $L^p(E)$ is the subspace of L^p functions which vanish a.e. off of E. χ is the identity function on C: $\chi(e^{ix}) = e^{ix}$.

Now let b be in the unit ball of H^∞. We will construct the Sz.-Nagy and Foiaş model which has b as its characteristic operator function. Let $\Delta = (1 - |b|^2)^{1/2}$ and $E = \{e^{ix}: \Delta(e^{ix}) > 0\}$. H denotes the Hilbert space $H^2 \oplus L^2(E)$ with the obvious norm. Elements of H will be written

$$(f, g) \quad \text{or} \quad \begin{pmatrix} f \\ g \end{pmatrix} \qquad \text{with } f \in H^2, \quad g \in L^2(E).$$

Let M be the subspace $\{(bf, \Delta f): f \in H^2\}$ and set $K = H \ominus M$. The orthogonal projection of H onto K is denoted by P. U is the operator on H given by $U(f, g) = (\chi f, \chi g)$; by S we mean the operator $S = PU|K$. S^* is the canonical model having b as its characteristic operator function. For convenience we work with S rather than S^*. It is well known [4], [7] that any completely nonunitary contraction T such that $I - T^*T$ and $I - TT^*$ have rank 1 is unitarily equivalent to an operator S of this type.

Now suppose that $\{b_j\}$ is a collection (indexed by integers $j = 1, 2, \cdots$) of two or more elements of the unit ball in H^∞. For each j, let Δ_j, E_j, H_j, M_j, K_j, U_j, P_j and S_j be to b_j as Δ, E, H, M, K, U, P and S are to b above. We denote by $\mathcal{G}_{ij}$ the set of all 2×2 matrix valued functions on C of the form

$$A = \begin{pmatrix} \phi & 0 \\ \theta & \psi \end{pmatrix} \quad \text{where } \phi \in H^\infty, \quad \theta \in L^\infty(E_j) \quad \text{and} \quad \psi \in L^\infty(E_i \cap E_j).$$

$\mathcal{G}_{ij}$ is a Banach space with norm $\|A\|_\infty = \operatorname{ess\,sup} \{\|A(e^{ix})\| : 0 \leqq x < 2\pi\}$, where $\|A(e^{ix})\|$ is the operator norm of the 2×2 matrix $A(e^{ix})$. We define a linear map $\Lambda_{ij}: \mathcal{G}_{ij} \to \mathcal{L}(H_i, H_j)$ by $\Lambda_{ij}(A)F = AF$, $A \in \mathcal{G}_{ij}$ and

$$F = \begin{pmatrix} f \\ g \end{pmatrix} \in H_i.$$

LEMMA 1. *Λ_{ij} is an isometric isomorphism of $\mathcal{G}_{ij}$ onto $\mathcal{J}(U_i, U_j)$.*

Let $\mathcal{A}_{ij}$ be the subset of $\mathcal{G}_{ij}$ consisting of all A of the form

$$\begin{pmatrix} \phi & 0 \\ \theta & \psi \end{pmatrix},$$

where

(a) $\phi \in H^\infty \cap b_j b_i^{-1} H^\infty$, and

(b) $\theta = b_j^{-1}\Delta_j\phi - b_i^{-1}\Delta_i\psi$.

A short computation shows that if $A \in \mathfrak{A}_{ij}$ and $B \in \mathfrak{A}_{jk}$, then $BA \in \mathfrak{A}_{ik}$. Also, $\mathfrak{A}_{ii}$ is a commutative algebra.

LEMMA 2. $\mathfrak{A}_{ij} = \{A \in \mathcal{G}_{ij}: \Lambda_{ij}(A)M_i \subset M_j\}$.

Let Γ_{ij}: $\mathfrak{A}_{ij} \to \mathcal{L}(K_i, K_j)$ be the linear map defined by $\Gamma_{ij}(A)F = P_j(AF)$, $A \in \mathfrak{A}_{ij}$ and $F \in K_i$. We denote by $\mathfrak{N}_{ij}$ the set of all $A \in \mathfrak{A}_{ij}$ of the form

$$A = \begin{pmatrix} b_j u & 0 \\ \Delta_j u & 0 \end{pmatrix} \qquad \text{where } u \in H^\infty.$$

LEMMA 3. (i) *If* $A \in \mathfrak{A}_{ij}$ *and* $B \in \mathfrak{A}_{jk}$, *then* $\Gamma_{ik}(BA) = \Gamma_{jk}(B)\Gamma_{ij}(A)$.

(ii) Ker $\Gamma_{ij} = \mathfrak{N}_{ij}$.

It is easily verified that Γ_{ij} takes values in $\mathfrak{I}(S_i, S_j)$. Using the lifting theorem [3], [8] and the previous lemmas, this statement can be strengthened in the following crucial way.

THEOREM 1. *The natural map* $\mathfrak{A}_{ij}/\mathfrak{N}_{ij} \to \mathfrak{I}(S_i, S_j)$ *induced by* Γ_{ij} *is isometric and onto.*

THEOREM 2. S_1 *and* S_2 *are similar if and only if*

(i) $b_1 b_2^{-1}$ *and* $b_2 b_1^{-1}$ *are in* H^∞, *and*

(ii) E_1 *and* E_2 *differ only by a Lebesgue null set.*

The proof of Theorem 2 rests on Lemma 3(i), Theorem 1 and the observation that S_1 and S_2 are similar provided $\mathfrak{I}(S_1, S_2)$ contains an invertible operator.

3. **An application to integral operators.** An operator A on a Hilbert space is *completely nonselfadjoint* if there is no nonzero reducing subspace N for A such that $A|N$ is selfadjoint. Every operator has a unique (possibly trivial) reducing subspace N such that $A|N$ is selfadjoint and $A|N^\perp$ is completely nonselfadjoint. $A|N$ and $A|N^\perp$ are, respectively, the selfadjoint and completely nonselfadjoint parts of A. If A has nonnegative imaginary part, $T = (A - i/2)(A + i/2)^{-1}$ is a contraction (see [7, p. 348]), $T|N$ is unitary and $T|N^\perp$ is completely nonunitary.

Now let α be a real, measurable, essentially bounded function on $[0, 1]$ and let $c \in L^2(0, 1)$. We associate with the pair (α, c) the operator A on L^2 (0, 1) given by

$$(1) \qquad (Af)(x) = \alpha(x)f(x) + ic(x)\int_0^x \overline{c(t)}f(t)dt.$$

In another paper by the author [6] it was shown that the completely nonunitary part of $T=(A-i/2)(A+i/2)^{-1}$ is unitarily equivalent to the operator S associated, as in §2, with the H^∞ function

$$(2) \quad b(z) = \exp\left\{(1-z)\int_0^1 \frac{1-\beta(x)}{\beta(x)-z}\,|c(x)|^2dx\right\}, \qquad |z| < 1,$$

where $\beta(x)=(\alpha(x)-i/2)(\alpha(x)+i/2)^{-1}$, $0\leqq x\leqq 1$.

Now suppose that for $i=1, 2$, A_i and b_i are associated with the pair (α_i, c_i) in the same way that A and b are associated with (α, c) in (1) and (2). Let S_i be associated with b_i as in §2. From the above discussion it is clear that the completely nonselfadjoint parts of A_1 and A_2 are similar if and only if S_1 and S_2 are similar. This observation plus Theorem 2 and some computation yields the following

THEOREM 3. *For $j=1, 2$, let μ_j be the measure on $(-\infty, \infty)$ given by $\mu_j(F)=\int_{\alpha_j^{-1}(F)}|c_j|^2dm$, where F is any Borel subset of $(-\infty, \infty)$ and m is Lebesgue measure. Let $d\mu_j=w_jdm+d\mu_{s,j}$ be the Lebesgue decomposition of μ_j, i.e., $0\leqq w_j\in L^1(-\infty, \infty)$ and $\mu_{s,j}$ is a singular measure. Then the completely nonselfadjoint parts of A_1 and A_2 are similar if and only if*

(i) $\mu_{s,1}=\mu_{s,2}$,

(ii) $\{x: w_1(x)=0\}$ *and* $\{x: w_2(x)=0\}$ *differ only by a Lebesgue null set,*

(iii) w_1-w_2 *is essentially bounded.*

A result in [6] characterizes those pairs (α, c) for which A is completely nonselfadjoint. From this and Theorem 3 we have the following:

COROLLARY 1. *If α_1 and α_2 are monotone and c_1, c_2 in $L^2(0, 1)$ vanish only on a set of measure zero, then A_1 and A_2 are similar if and only if conditions* (i), (ii) *and* (iii) *of Theorem* 3 *hold.*

4. **An application to de Branges-Rovnyak theory.** Suppose b is in the unit ball of H^∞. $\mathcal{H}(b)$ denotes the set of all H^2 functions f with the property that

$$(3) \qquad \sup\{\|f+bg\|_2^2 - \|g\|_2^2 : g \in H^2\} < \infty.$$

It is well known (see [1], [2]) that $\mathcal{H}(b)$ is a Hilbert space with the norm $\|f\|_b$ given by the square root of the supremum (3). If

$(Qf)(e^{ix}) = e^{-ix}(f(e^{ix}) - f(0))$, Q defines a contraction operator on $\mathcal{H}(b)$. In the work of de Branges and Rovnyak, operators of the form Q provide a canonical model for contractions T having no isometric restriction and with rank $(I - T^*T) = \text{rank } (I - TT^*) = 1$. It is not hard to show that Q is a model for such a T if $\log \Delta \notin L^1$.

Suppose that $b_i \in H^\infty$ and $\|b_i\|_\infty \leqq 1$ $(i = 1, 2)$. If, as sets, $\mathcal{H}(b_1) = \mathcal{H}(b_2)$, the inclusion maps i_1: $\mathcal{H}(b_1) \to \mathcal{H}(b_2)$ and i_2: $\mathcal{H}(b_2) \to \mathcal{H}(b_1)$ are bounded and $i_2 = i_1^{-1}$. If Q_i is related to b_i as Q is related to b, it is clear that $i_1 Q_1 = Q_2 i_1$ so Q_1 and Q_2 are similar. J. Rovnyak posed the question of whether equality of $\mathcal{H}(b_1)$ and $\mathcal{H}(b_2)$ is a stronger condition than similarity of Q_1 and Q_2. The answer, which follows from Theorem 2, is yes.

THEOREM 4. *Suppose that* $\log \Delta_i \notin L^1$ $(i = 1, 2)$. *In order that* $\mathcal{H}(b_1) = \mathcal{H}(b_2)$ *(as sets) it is necessary and sufficient that*

(i) $b_1 b_2^{-1}$ *and* $b_2 b_1^{-1}$ *lie in* H^∞, *and*

(ii) *there exist constants* C, $D > 0$ *such that* $C\Delta_1 \leqq \Delta_2 \leqq D\Delta_1$ *a.e.*

Q_i^* is unitarily equivalent to the operator S_i of Theorem 2, so conditions (i) and (ii) of Theorem 2 are necessary and sufficient for Q_1 and Q_2 to be similar.

Details and proofs will appear elsewhere.

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