BOOK REVIEWS

The variational theory of geodesics by M. M. Postnikov. Translated from the Russian by Scripta Technica, Inc. Edited by Bernard R. Gelbaum. Saunders, Philadelphia, Pa., 1967. 200 pp. $6.00.

Publishers have an understandable tendency to suggest that their products are suitable for large audiences. It is claimed on the cover of the volume reviewed here that it "can be effectively studied outside the discipline of the classroom" and that it "is readily understandable to those with a solid grounding in calculus." The publisher's optimism is probably based on Postnikov's tendency to include details that many writers for mature audiences would omit. It is indeed possible that a bright student who knows calculus well would be able to follow most of the proofs; but following individual proofs is not the same thing as reading a book like this one. The internal evidence suggests that Postnikov had a much more knowledgeable reader in mind since he includes almost no exercises, examples, motivation, historical orientation, or indication of how his subject is related to the rest of mathematics. A more plausible choice for a suitable reader would be someone (most likely a budding topologist) who already knows why he wants to study the Morse theory of geodesics and what it is good for, but who knows no differential geometry.

The book falls into two parts, the first of which consists of Chapters I, II, and III and is devoted to more or less standard topics in differential geometry. It begins with the definition of a manifold, develops most of the usual facts about tensor fields, connections, geodesics, curvature, and Riemannian geometry and ends with the Hopf-Rinow-Myers theorem which is (typically) not identified by name.

Except for a few minor twists everything covered in these chapters is well known and can be found in many other places. One would therefore expect that they were written with a stack of other sources close at hand. Unless there have been incredibly many coincidences, *Differential geometry and symmetric spaces* by S. Helgason must have been on the top of the pile most of the time. The basic approach is like Helgason's in its emphasis on tensor fields as modules over the C^∞ functions rather than as sections of vector bundles. Moreover, many of the sections appear to be mere translations of Helgason's rather precise and formal exposition into Postnikov's much more casual style. The similarities become quite evident with the C^∞ Urysohn lemma (Helgason p. 6, Postnikov p. 7). From there on, almost everything in Postnikov except for a few digressions, has a counterpart in

Helgason, which is often almost identical modulo the differences in style. Some typical examples are provided in the discussion of vector fields (Helgason p. 9, Postnikov p. 12), the fundamental theorem of Riemannian geometry (Helgason p. 48, Postnikov pp. 80–82), the convergence lemma (Helgason p. 56, Postnikov p. 88), and the Hopf-Rinow-Myers theorem (Helgason pp. 66–67, Postnikov pp. 102–103). Postnikov has also followed Helgason in using the Cartan structural equations in order to get the local minimizing property of geodesics and in my view this does not work out very well. The trouble is that Postnikov has avoided the use of the exterior derivative, hence his derivation of the Cartan equation is somewhat tedious. In addition, he makes no use of the equations after the minimizing property has been derived.

The large number of similarities between Postnikov's and Helgason's expositions seem to rule out the possibility of coincidence and to make it clear that Postnikov was, to say the least, strongly influenced by Helgason's book. There is perhaps nothing reprehensible in this, although it raises some question as to whether Postnikov's book has fulfilled a burning need. What seems entirely inexcusable is that there is no acknowledgement of even the existence of Helgason's book. Indeed, one of the major defects of Postnikov's book is that it contains no references of any kind to any other book or article.

The last two chapters are aimed at proving the Morse index theorem, the Bott reduction theorem and at discussing their generalizations in which focal points replace conjugate points. In the first part of Chapter IV some needed facts about Jacobi fields and conjugate points are derived. The development here seems to have been influenced by Zisman's lectures in Séminaire Henri Cartan 12, although there is, of course, no acknowledgement of this.

With all due respect to the theorems of Morse and Bott, it seems fair to say that much of their beauty and importance is not intrinsic but stems from their applications. Unfortunately, Postnikov does not even hint these theorems are good for anything. For example, the Bott theorem asserts, among other things, that a certain space of paths has the same homotopy type as a manifold on which a Morse function is defined, but there is no suggestion that Morse functions are useful for investigating homotopy type. Since there are no references to works that would shed any light on the uses of the main theorems of the book, the author surely was thinking in terms of a knowledgeable or at least a well advised reader.

There remains some question of how adequate the book is even for the most ideal reader. In the first place, the translators and printers have not always been careful, so there are a large number of minor errors (e.g., "∇_X is a differentiation of . . ." instead of "∇_X is a derivation of . . ." p. 65 or the placement of parentheses in the formula at the bottom of p. 172). More serious confusion is caused by translating such as "everywhere differentiable" where surely "infinitely differentiable" is meant (p. 1) or "it will be sufficient to assume . . ." instead of "it will be sufficient to prove . . ." (p. 38). There is an occasional sentence like: "Any function $p \rightarrow \omega_p$ that assigns to each point $p \in M$ a convector (sic) $\omega_p \in \sigma_1(p)$ and satisfies this smoothness condition satisfies some linear differential form ω." (p. 25), which seems impenetrable until one locates the mistranslated word.

The author's informal style makes much of the reading easy and pleasant, but it also has its drawbacks. Very few assertions are identified as theorems, propositions, lemmas, etc. Instead the reader is presented with a series of italicized statements which range from very trivial (e.g., p. 132) to very important (e.g., p. 149) and he is left to supply his own emphasis. Sometimes (e.g., pp. 128–129) a second statement occurs and is proved in the midst of the proof of the first one. This is a perilous procedure, at best, and when as in this book no device like "q.e.d." is used to indicate the end of a proof, the result can be disastrous.

There are some difficulties which go beyond those of printing, translation, format, and style. The most serious errors occur in the assertions and alleged proofs on pp. 101–102 that in a complete Riemannian manifold M the distance function $\rho: M \times M \rightarrow R$ is smooth (i.e., C^∞) at every point (p, q) where $p \neq q$, and ρ^2 is smooth everywhere on $M \times M$. A moment's reflection on the case $M = S^1$ makes it clear that these assertions are hopelessly wrong. The context makes it clear that these errors cannot be explained away on the grounds that some hypothesis was omitted by a careless printer. This sort of mistake seems unacceptable in a book whose purpose is largely to shed light on the nature of conjugate points.

The exposition is usually clear, but there are some foggy passages. I never understood why there were two subscripts on the A's in the first display on p. 132. On p. 139, $Q_\theta(z)$ is defined at the top and after a page of argument, the definition is triumphantly recovered as formula (1). Moreover, formula (1) is subsequently referred to in several places with the implication that it has some content and is not merely the definition of $Q_\theta(z)$.

Writing a book is a serious endeavor. My guess is that Postnikov had no intention of writing one at all, but rather prepared a set of informal lecture notes which an American publisher has mediocrely translated and tried to palm off on the public. It seems unlikely that there will be many who will want to read the resulting volume when most of the same material and more is covered so well in J. Milnor's *Lecture notes on Morse theory.*

RICHARD SACKSTEDER

Singularities of smooth maps by James Eells, Jr. Gordon and Breach, New York, 1967. 104 pp. $5.50; paper $3.00.

This book is a reprinting of a set of lecture notes for the first half of a course given by James Eells in about 1960. The notes have essentially not been reworked and so maintain—as the author mentions in his preface—an "incomplete and definitely temporary character." The book is quite elementary and consists of three chapters. The first two require nothing more than calculus of several variables while the third uses a little algebraic topology.

The first chapter is a quick review of calculus of several variables leading to the definitions surrounding the notion of a finite dimension manifold (including the tangent bundle). The existence of the globalizing tool, the partition of unity, is proved completely.

The second chapter begins the study of singularities of smooth maps of compact manifolds with Whitney's theorems giving the open-density of imbeddings (immersons) among all C^k maps of an n-manifold into $\mathbf{R}^{2n+1}$ ($\mathbf{R}^{2n}$). The weak form of the C^∞ Sard-Dubovitsky-Morse theorem—that a C^∞ map takes its critical set into a meager subset of the target—is proved and applied to show that most immersions of compact n-manifolds in $\mathbf{R}^{2n}$ have only clean self-intersections—thus only isolated double points and no triple points.

The two simplest cases of maps which typically display some singular behavior are now discussed—maps of n-manifolds into $\mathbf{R}^{2n-1}$ and into $\mathbf{R}$. For each, the usual notion of nondegenerate singularity is defined in terms of local coordinates, local normal forms are given and the *generic* maps, those having only nondegenerate singularities, are shown to fill an open dense subset of the C^k-maps.

The general question of the existence of an open dense set of "generic" maps in $C^k(X, Y)$ is posed and some of the formalism of jets is introduced. Unfortunately the author does not develop quite enough of it to state the general transversality theorem of Thom, and so cannot even suggest that everything in the chapter except the Sard-type theorem is a corollary of this result.

The main object of the last chapter is the proof of the Morse-Pitcher inequalities which relate the critical points of a generic real-valued function on a compact manifold with the betti-numbers and