SPECTRAL DECOMPOSITION OF ERGODIC FLOWS ON L^p[1]

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Let M be a totally σ-finite measure space and U_s (s real) be a one parameter group of measure—preserving transformations of M satisfying appropriate measurability and continuity conditions. We let $U_s: L^p(M)\to L^p(M)$ by $U_s f = fU_s$. If $p=2$ Stone's spectral theorem for unitary operators [2] says that there is a spectral family of projections $E_\lambda: L^2(M)\to L^2(M)$ such that for $f\in L^2(M)$

$$U_s f = \int_{-\infty}^{\infty} e^{2\pi i\lambda s} dE_\lambda f \tag{1}$$

from which we show that if $\psi\in L^1(R)$ and $\hat{\psi}$ is the Fourier transform of ψ,

$$\int_{-\infty}^{\infty} \hat{\psi}(\lambda) dE_\lambda f = \int_{-\infty}^{\infty} \psi(s) U_s f ds. \tag{2}$$

We will say that a function is normalized at its jumps if it has only jump discontinuities and the value at each jump is the average of the values from the sides. Let χ_τ be the normalized characteristic function of $(-\infty, \tau]$. We approximate χ_τ pointwise with the Fourier transforms of L^1 functions and use (2) to show for $f\in L^2(M)$, $D_\lambda f = E_{\lambda-0}f + E_\lambda f - f$ where

$$D_\lambda f = \frac{-1}{i\pi}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{s} e^{-2\pi i\lambda s} U_s f ds \tag{3}$$

and so $E_\lambda f = f + \frac{1}{2}D_\lambda f - \frac{1}{2}D_\lambda^2 f$.

A slight modification of a theorem in [1] shows that D_λ is a bounded transformation on $L^p(M)$ $(1<p<\infty)$ with the bound independent of λ. This gives

THEOREM 1. *D_λ and hence E_λ extend from $L^p(M)\cap L^2(M)$ to $L^p(M)$ by continuity. For $f\in L^p(M)$, $\|E_\lambda f\|_p$ is bounded uniformly in λ. $E_{\lambda+0}f = E_\lambda f$. $E_\lambda E_\tau f = E_\lambda f$ if $\lambda \leqq \tau$. $\|E_\lambda f\|_p \to 0$ as $\lambda\to -\infty$. $\|E_\lambda f - f\|_p \to 0$ as $\lambda \to +\infty$.*

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The theorems from [1] also show that convergence of the symmetrically truncated integrals to the integral in (3) is dominated and pointwise a.e.

We show that $(E_\lambda f, g)$ is a continuous function of λ except for a set of jumps which is countable (and does not depend on f, g). Thus we can form the Stieltjes integral of any absolutely continuous function with respect to $(E_\lambda f, g)$ over a bounded interval. In particular we can integrate $e^{2\pi i\lambda s}$ over a bounded interval. For f and g in $L^2(M)$

$$(4) \qquad ((E_b - E_a)U_s f, g) = \int_a^b e^{2\pi i\lambda s}(U_s f, g)ds.$$

We assume from now on $f \in L^p(M)$, $g \in L^{p'}(M)$, $1/p + 1/p' = 1$, $1 < p < \infty$. The absolute value of the integral in (4) is no bigger than $(b-a)\|f\|_p\|g\|_{p'}$ so the integral is a continuous function of f and g. So is the left side of (4). Hence (4) holds for $f \in L^p(M)$, $g \in L^{p'}(M)$. Letting $a \to -\infty$ and $b \to \infty$ we get (1) for $L^p(M)$ where the integral in (1) may be taken to be a weak integral.

We now define a slight generalization of the Stieltjes integral. Suppose h has support in $[a, b]$ and is continuous from the right and has a limit from the left everywhere, and suppose $\Lambda_\epsilon = \{\lambda \in [a, b] \mid |h(\lambda) - h(\lambda - 0)| > \epsilon\}$ is finite for each $\epsilon > 0$ (for example $h(\lambda) = (E_\lambda f, g)$). If α is of bounded variation on $[a, b]$ then the integral of h with respect to α exists in the following sense: For $\epsilon > 0$ we will only consider partitions $P \supset \Lambda_\epsilon$. If $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ let $S_P = \sum_1^n h(\xi_j)[\alpha(x_j) - \alpha(x_{j-1})]$ where $x_{j-1} < \xi_j < x_j$. For such partitions and for $\epsilon > 0$, there exists l depending only on ϵ, h and α such that $|S_P - S_{P'}| < \epsilon$ whenever mesh $P < l$ and mesh $P' < l$.

We use the above integral and some lemmas to show

THEOREM 2. *Let θ_j be Fourier multipliers for $L^p(R)$ with multiplier norms M_j. Assume θ_j is normalized at its jumps and has bounded variation locally.*

$$(5) \qquad (A(\theta_j)f, g) = \int_{-\infty}^{\infty} \theta_j(\lambda)d(E_\lambda f, g)$$

exists as the limit of the truncated integrals and $|(A(\theta_j)f, g)| \leq M_j\|f\|_p\|g\|_{p'}$.

$$(6) \qquad (A(\theta_1) \circ A(\theta_2)f, g) = \int_{-\infty}^{\infty} \theta_1(\lambda)\theta_2(\lambda)d(E_\lambda f, g)$$

i.e. $A(\theta_1) \circ A(\theta_2) = A(\theta_1 \cdot \theta_2)$.

THEOREM 3. *Suppose θ is a multiplier for $L^p(R)$, and θ_j, M_j are as* *in Theorem* 2 *and* $\theta_j \to \theta$ *pointwise and there exists* M *such that* $|\theta_j(\lambda)| \leq M$, $M_j \leq M$ *for all* j, λ. *Then* $(A(\theta_j)f, g) \to (A(\theta)f, g)$.

THEOREM 4. *If* ϕ *is zero except at* $t_1 \cdots t_n \cdots$ *and* $\sum_1^\infty |\phi(t_j)| < \infty$ *then*

$$\int_{-\infty}^{\infty} \phi(\lambda) d(E_\lambda f, g) = \sum_{j=1}^{\infty} \phi(t_j)[(E_{t_j} f, g) - (E_{t_j - 0} f, g)]. \tag{7}$$

These theorems allow us to integrate many multipliers with respect to $(E_\lambda f, g)$.

We now construct two complex semigroups. For $y \neq 0$ let

$$(T^n_{x,y} f, g) = \frac{-1}{2\pi i} \int_{-n}^{n} \frac{1}{s + iy} (U_{x-s} f, g) ds - \frac{1}{2} ((E_0 - E_{-0}) f, g). \tag{8}$$

Temporarily let us assume $f \in L^p(M) \cap L^2(M)$, $g \in L^{p'}(M) \cap L^2(M)$ and apply (2) to get

$$(T^n_{x,y} f, g) = \int_{-\infty}^{\infty} \theta^n_y(\lambda) d(E_\lambda f, g) - \frac{1}{2} ((E_0 - E_{-0}) f, g) \tag{9}$$

where

$$\theta^n_y(\lambda) = \int_{-n}^{n} e^{2\pi i \lambda s} \frac{1}{s + iy} ds. \tag{10}$$

We see from (8) that $(T^n_{x,y} f, g)$ is a continuous function of $f \in L^p(M)$, $g \in L^{p'}(M)$ for each x, y, n $(y \neq 0)$. We show that the right side of (9) is continuous in f and g and has a limit as $n \to \infty$ by showing that $\theta^n_y(\lambda)$ and $\theta_y(\lambda) = \lim \theta^n_y(\lambda)$ satisfy the hypotheses of Theorems 2 and 3. To see this we subtract the truncated (at 1 and n) Hilbert transform from the truncated kernals $1/(s+iy)$. Thus $(T_{x,y} f, g) = \lim (T^n_{x,y} f, g)$ exists. We show that $T_{x,y} \circ T_{x',y'} = T_{x+x',y+y'}$ and that $T_{x,y}$ is an analytic function of $z = x + iy$.

$\mathrm{Im}(E_0 - E_{-0})$ is the set of functions h such that $U_s h = h$ for all s. We will assume from now on $f \in \mathrm{Ker}(E_0 - E_{-0})$.

There is an equation for $T_{x,y}$ like the equation for $T^n_{x,y}$ in (9). From this we show that if $y > 0$ and $f \in \mathrm{Ker} E_0$, $T_{x,-y} f = 0$ so

$$T_{x,y} f = T_{x,y} f - T_{x,-y} f = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{s^2 + y^2} U_{x-s} f ds. \tag{11}$$

For $y < 0$ $T_{x,y} f = 0$.

Similarly for $f \in \mathrm{Im} E_0$, (11) holds if $y < 0$ and $T_{x,y} f = 0$ if $y > 0$.

For $y>0$ and $f\in\operatorname{Ker}E_0$ or $y<0$ and $f\in\operatorname{Im}E_0$ write $T'_{x,y}f(\xi)$ for the integral at the right in (11) evaluated at $\xi\in M$. Since (11) holds in $L^p(M)$ we have for each x, y $T_{x,y}f(\xi)=T'_{x,y}f(\xi)$ for almost all $\xi\in M$ but the set where $T_{x,y}f(\xi)\neq T'_{x,y}f(\xi)$ depends on (x, y). We show that there is a set $M_f\subset M$ such that measure $(M-M_f)=0$ and $T'_{x,y}f(\xi)$ converges absolutely for all $\xi\in M_f$ and all x, y $(y\neq 0)$. M_f does not depend on x, y. In Theorems 5 and 6 assume $f\in\operatorname{Ker}(E_0-E_{-0})$.

THEOREM 5. *The maximal function* $Sf(\xi)=\operatorname{Sup}\{|T'_{x,y}f(\xi)|\mid(x,y)$ *is in a cone not tangent to the line* $y=0\}$ *is of type* (p, p) $(1<p<\infty)$.

THEOREM 6. *For* $f\in\operatorname{Im}E_0$, $T'_{x,y}f\rightarrow U_{x_0}f$ *as* $(x, y)\rightarrow(x_0, 0)$ *nontangentially from below.*

For $f\in\operatorname{Ker}E_0$, $T'_{x,y}f\rightarrow U_{x_0}f$ *as* $(x, y)\rightarrow(x_0, 0)$ *nontangentially from above.*

For $h\in L^p(M)$, $T_{x,y}h\rightarrow(E_0-E_{-0})h$ *as* $y\rightarrow\infty$ *and* x *remains in any bounded set. Convergence above is* L^p *convergence, dominated and pointwise convergence on a subset of* M *having full measure.*

The first two pieces of Theorem 6 say that the original group is a sort of direct sum of the two analytic semigroups we constructed.

REFERENCES

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