TOPOLOGICAL CLASSIFICATION OF INFINITE DIMENSIONAL MANIFOLDS BY HOMOTOPY TYPE[1]

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1. **Introduction.** In this paper we prove that if M and N are connected paracompact manifolds modeled on a normed TVS, F, such that F is homeomorphic ($\cong$) to F^ω (countably infinite product of F),[3] then M and N are homeomorphic if and only if they have the same homotopy type. We also prove that if M and N are connected paracompact manifolds modeled on a metrizable locally-convex (MLC) TVS, $F \cong F^\omega$, then each map $f: M \to N$ can be approximated by a closed embedding $g: M \to N$ and an open embedding $h: M \to N$ such that $f \sim g \sim h$ (homotopic). These and other results will be proved on the basis of results in recent, not yet published, papers written separately by the authors. See [5], [6], and [7]. These results already have been proved for separable Fréchet spaces by several authors, see [4] for references.

2. **Theorems to quote.** By manifold we will always mean a paracompact manifold. By TVS we mean a Hausdorff topological vector space.

S1. THEOREM [7]. *If M is a manifold modeled on a metrizable* TVS, *$F \cong F^\omega$, then $M \times F \cong M$.*

Let X and Y be spaces, $\mathcal{U}$ be an open cover of Y, and $f, g: X \to Y$. Then f and g are said to be $\mathcal{U}$-approximate if for each $x \in X$ there is a

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[3] *Added in proof.* This condition is satisfied by all infinite-dimensional Hilbert spaces, reflexive Banach spaces, and separable Fréchet spaces and is not known to be false for any Fréchet space. (See Bessaga and Kadec, *On topological classification of* (to appear).)

$U \in \mathfrak{U}$ that contains each of $f(x)$ and $g(x)$. If C is a collection of functions from X to Y, then f is said to be *approximated* by members of C if for each open cover $\mathfrak{U}$ of Y, there exists $h \in C$ such that f and h are $\mathfrak{U}$-approximate.

S2. THEOREM [7]. *Let M be an open subset of a metrizable* LCTVS, $F \cong F^\omega$. *Then the projection map p_1: $M \times F \to M$ can be approximated by homeomorphisms H: $M \times F \to M$ such that $H \sim p_1$.*

H1. THEOREM [5]. *A microbundle is trivial if*

(a) *its fiber is a* TVS, *F such that $F \cong F^\omega$, and*

(b) *its base is a paracompact space with homotopy type of a simplicial (or CW-) complex.*

H2. THEOREM [6]. *If M is a connected manifold modeled on a metrizable* TVS, *$F \cong F^\omega$, then M can be embedded as a closed subset of F.*

H3. THEOREM [6]. *Let M and N be manifolds modeled on a* MLCTVS, *$F \cong F^\omega$. If $h: M \to N$ is a closed embedding, then there is an open embedding $g: M \times F \to N \times F$ such that $g(m, 0) = (h(m), 0)$ for each $m \in M$.*

H4. THEOREM [6]. *Let N be a manifold modeled on a normed* TVS, *$F \cong F^\omega$, and let X be an* ANR *(for metric spaces). If f, g: $X \to N$ are homotopic closed embeddings, then there is an invertible isotopy h: $(N \times F) \times I \to (N \times F) \times I$ such that $h(n, y, 0) = (n, y, 0)$ for each $n \in N$ and $y \in F$ and $h(f(x), 0, 1) = (g(x), 0, 1)$ for each $x \in X$.*

H4 is a crucial step in proving

H5. THEOREM [6]. *Let M and N be connected manifolds modeled on a normed* TVS, *$F \cong F^\omega$. If $f: M \to N$ is a homotopy equivalence, there exists a homeomorphism $h: M \times F \to N \times F$ such that $h \sim f \times$*id.

3. Theorems to prove. For each of the following theorems let M and N be connected paracompact manifolds modeled on a MLCTVS, $F \cong F^\omega$.

A. THEOREM. *The manifold M can be embedded as an open subset of F.*

COROLLARY. *The projection $p_1: M \times F \to M$ can be approximated by homeomorphisms $h: M \times F \to M$ such that $h \sim p_1$.*

B. THEOREM. *Each map $f: M \to N$ can be approximated by closed embeddings $h_1: M \to N$ and open embeddings $h_2: M \to N$ such that $f \sim h_1 \sim h_2$.*

C. THEOREM. *If F is a normed* TVS, *then each homotopy equivalence between M and N is homotopic to a homeomorphism.*

Following R. D. Anderson we say that a subset K of a space X has *Property* Z in X if, for each nonempty, homotopically-trivial open set $U\subset X$, $U-K$ is nonempty and homotopically-trivial. The following theorem was proved for separable Fréchet manifolds in [1] and for special cases for nonseparable manifolds in [2].

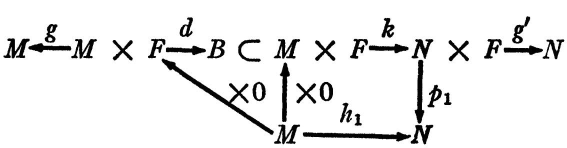
D. THEOREM. *If K is a closed set with Property* Z *in M, then K is negligible, that is, $M-K$ is homeomorphic to M. In fact, the homeomorphism is homotopic to the inclusion, $M-K\to M$.*

4. **Proofs.** If $\mathcal{U}$ and $\mathcal{V}$ are collections of subsets of a given set X, then to say that $\mathcal{V}$ *refines* $\mathcal{U}$ means that each element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$. Denote this by $\mathcal{V}<\mathcal{U}$. If $U\subset X$, let $\mathrm{St}(U,\mathcal{V})=\bigcup\{V\in\mathcal{V}: U\cap V\neq\emptyset\}$ and let $\mathrm{St}(\mathcal{U},\mathcal{V})=\{\mathrm{St}(U,\mathcal{V}): U\in\mathcal{U}\}$.

Let X and Y be spaces, $\mathcal{U}$ and $\mathcal{V}$ be open covers of Y, and f, g, h: $X\to Y$. Then, f and g are $\mathcal{U}$-approximate if $\{\{f(x), g(x)\}: x\in X\}<\mathcal{U}$. Denote this by $\{f, g\}<\mathcal{U}$. If, in addition $\{g, h\}<\mathcal{V}$, then it follows that $\{f, h\}<\mathrm{St}(\mathcal{U}, \mathcal{V})$. Also denote $\{f^{-1}(U): U\in\mathcal{U}\}$ by $f^{-1}(\mathcal{U})$, $\{U\times F: U\in\mathcal{U}\}$ by $\mathcal{U}\times F$, and $\{\mathrm{Cl}\ U: U\in\mathcal{U}\}$ by $\mathrm{Cl}\ \mathcal{U}$.

PROOF OF THEOREM A. It follows immediately from H2, H3, and S1.

PROOF OF THEOREM B. Let $\mathcal{U}$ be an open cover of N. We can assume that N is an open subset of F and that each element of $\mathcal{U}$ is convex. By standard shrinking techniques we may find open covers $\mathcal{V}$ and $\mathcal{W}$ of N such that $\mathrm{Cl}\ \mathcal{W}<\mathcal{V}<\mathrm{St}(\mathrm{St}(\mathcal{V}, \mathcal{W}), \mathcal{W})<\mathcal{U}$. By S2 take a homeomorphism $g': N\times F\to N$ that is $\mathcal{W}$-approximate to p_1. By H2 let j be a closed embedding of M into F. Then $h_1: M\to N$ defined by $h_1=g'\cdot(f, j)$ is a closed embedding of M into N and $\{f, h_1\}<\mathcal{W}$. Now apply H3 to $h_1: M\to N$ to obtain an open embedding $k: M\times F\to N\times F$. By S2 there is a homeomorphism $g: M\times F\to M$ such that $\{p_1, g\}<h_1^{-1}(\mathcal{W})$. If $W\in\mathcal{W}$, let $V(W)\in\mathcal{V}$ such that $\mathrm{Cl}\ W\subset V(W)$. Let $B=\{(x, y)\in M\times F$: if $x\in h_1^{-1}(W)$ for $W\in\mathcal{W}$, then $k(x, y)\in V(W)\times F\}$. Then B is open in $M\times F$ and contains $M\times\{0\}$.



One can construct an open embedding $d: M\times F\to B$ (for example see [5, Lemma 1.2]) such that $p_1=p_1\circ d$ and the above diagram commutes.

Define $h_2: M\to N$ by $h_2=g'\circ k\circ d\circ g^{-1}$. Since $\{g, p_1\}<h_1^{-1}(\mathcal{W})$ we

have $\{g \circ g^{-1}, p_1 \circ g^{-1}\} < h_1^{-1}(\mathcal{W})$ which is the same as $\{p_1 \circ \times 0, p_1 \circ g^{-1}\} < h_1^{-1}(\mathcal{W})$ since $p_1 \circ \times 0 = \mathrm{id} = g \circ g^{-1}$ and this means that $\{\times 0, g^{-1}\} < h_1^{-1}(\mathcal{W}) \times F$. Since $p_1 \circ d = p_1$ we have $\{d \circ \times 0, d \circ g^{-1}\} < h_1^{-1}(\mathcal{W}) \times F$ and then by the definition of B we have $\{k \circ d \circ \times 0, k \circ d \circ g^{-1}\} < \mathcal{V} \times F$ or equivalently $\{p_1 \circ k \circ d \circ \times 0, p_1 \circ k \circ d \circ g^{-1}\} < \mathcal{V}$. Since $\{p_1, g'\} < W$ we have $\{p_1 \circ k \circ d \circ g^{-1}, g' \circ k \circ d \circ g^{-1}\} < \mathcal{W}$ and hence $\{h_1, h_2\} < \mathrm{St}(\mathcal{V}, \mathcal{W})$ since $h_1 = p_1 \circ k \circ d \circ \times 0$ and $h_2 = g' \circ k \circ d \circ g^{-1}$. We also have $\{f, h_1\} < \mathcal{W} < \mathcal{U}$ and hence $\{f, h_2\} < \mathrm{St}(\mathrm{St}(\mathcal{V}, \mathcal{W}), \mathcal{W}) < \mathcal{U}$. Thus each of h_1 and h_2 is $\mathcal{U}$-approximate to f. Since each element of $\mathcal{U}$ is convex, it is clear that $f \sim h_1 \sim h_2$. □

PROOF OF THEOREM C. Let $f: M \to N$ be a homotopy equivalence. By H5 there exists a homeomorphism $h: M \times F \to N \times F$ such that $h \sim f \times \mathrm{id}$. By S2 there are homeomorphisms $g: M \times F \to M$ and $g': N \times F \to N$ such that $g \sim p_1$ and $g' \sim p_1$. Then $g' \circ h \circ g^{-1}$ is a homeomorphism of M onto N and $g' \circ h \circ g^{-1} \sim p_1 \circ h \circ g^{-1} \sim p_1 \cdot (f \times \mathrm{id}) \circ g^{-1} = f \circ p_1 \circ g^{-1} \sim f \circ g \circ g^{-1} \sim f$. □

PROOF OF THEOREM D. It follows easily from Eells and Kuiper [3] that the inclusion $M - K \to M$ is a homotopy equivalence and thus by Theorem C the inclusion is homotopic to a homeomorphism. □

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