ON BIEBERBACH EILENBERG FUNCTIONS

BY DOV AHARONOV

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I. **Introduction.** In this paper we bring the following two results: *Suppose that* $F(z)=b_1z+b_2z^2+,\cdots$ *is a B.E. function* (*i.e.* $F(z)$ *is regular in the unit circle*, $F(z)F(\zeta)\neq 1$ *for any* $|z|, |\zeta|<1$ *and* $F(0)=0$). *Then we have*

$$\sum_{k=1}^{\infty} |b_k|^2 \leqq 1. \tag{1}$$

This result contains, of course, the result

$$|b_n| \leqq 1, \qquad n = 1, 2, \cdots \tag{2}$$

which was conjectured by Rogosinsky [8] and was solved about ten years later by Lebedev and Milin [5].

The second result deals with univalent B.E. function $F(z)=b_1z+b_2z^2+\cdots$. For such function we have the following

$$|b_n| \leqq e^{-c/2}(n-1)^{-1/2}, \qquad n = 2, 3, \cdots, \tag{3}$$

where c is Euler constant.

This result is sharp in order of magnitude and the constant cannot be improved to be better than $e^{-1/2}$.

II. **The results of Lebedev and Milin.** Lebedev and Milin found [6], [7] some important results concerning coefficients of exponential functions which we quote here.

LEMMA 1. *Let* $A_1, A_2, A_3, \cdots$ *be an infinite sequence of arbitrary complex numbers such that* $\sum_{k=1}^{\infty} k|A_k|^2<\infty$. *Then for* $\exp\sum_{k=1}^{\infty} A_kz^k = \sum_{k=0}^{\infty} D_kz^k$ *we have*

$$\sum_{k=0}^{\infty} |D_k|^2 \leqq \exp \sum_{k=1}^{\infty} k|A_k|^2 \tag{4}$$

with equality only in the case $A_k=\rho^k\eta^k/k$, $k=1, 2, \cdots$ *where* $0\leqq\rho<1$ $|\eta|=1$.

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LEMMA 2. *Let* $\{A_k\}$ *and* $\{D_k\}$ *be defined as in Lemma* 1 (*without the limitation* $\sum_{k=1}^{\infty} k|A_k|^2 < \infty$). *Then*

$$(5)\qquad |D_n|^2 \leqq \exp\left(\sum_{k=1}^{n} k|A_k|^2 - \sum_{k=1}^{n} 1/k\right), \qquad n = 1, 2, \cdots$$

with equality only in the case $A_k = \eta^k/k$ *for* $k = 1, 2, \cdots, n$ *and* $|\eta| = 1$.

III. **Schiffer-Garabedian inequalities.** We quote here a theorem of Garabedian and Schiffer [1]:

LEMMA 3. *Suppose that* $F(z)$ *is a univalent B.E. function. Then we have for*

$$(6)\qquad \log \frac{F(z) - F(\zeta)}{(z - \zeta)[1 - F(z)F(\zeta)]} = \sum_{n,m=0}^{\infty} \gamma_{nm} z^n \zeta^m,$$

$$(7)\qquad \operatorname{Re}\left\{\sum_{n,m=0}^{N} \lambda_n \lambda_m \gamma_{nm}\right\} \leqq \sum_{n=1}^{N} \frac{|\lambda_n|^2}{n}$$

where $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_N$ *is a finite sequence of complex constants with* λ_0 *real.*

This remarkable result was proved first in [1] by variational methods. Later the result was proved in [3] by area methods. We note that in [1] the result was formulated in a different manner.

IV. **Coefficients of B.E. functions.** From Lemma 3 we deduce immediately the following:

$$(8)\qquad \sum_{k=1}^{\infty} k|\gamma_{k0}|^2 \leqq \log \frac{1}{|F'(0)|^2}.$$

(Indeed from Lemma 3 we have

$$\lambda_0^2 \operatorname{Re}\{\log F'(0)\} + 2\lambda_0 \operatorname{Re}\left\{\sum_{n=1}^{N} \lambda_n \gamma_{n0}\right\} \leqq 2 \sum_{n=1}^{N} \frac{|\lambda_n|^2}{n}.$$

By substitution $\lambda_0 = 2$, $\lambda_n = n\bar{\gamma}_{n0}$ we get (8).)

We are now in a position to prove

THEOREM 1. *Let* $F(z) = b_1 z + b_2 z^2 + \cdots$ *be a B.E. function; then* (1) *follows.*

PROOF. By substituting $\zeta = 0$ in (6) we have

$$(9)\quad \log \frac{F(z)}{z} = \sum_{n=0}^{\infty} \gamma_{n0} z^n, \quad \frac{F(z)}{zF'(0)} = \exp\left(\sum_{k=1}^{\infty} \gamma_{0k} z^k\right) = \sum_{k=1}^{\infty} \frac{b_k}{F'(0)} z^{k-1}.$$

By Lemma 1 and (8) we get

$$\sum_{k=1}^{\infty} \frac{|b_k|^2}{|F'(0)|^2} \leqq \exp\left(\sum_{k=1}^{\infty} k\,|\gamma_{0k}|^2\right) \leqq \frac{1}{|F'(0)|^2}. \tag{10}$$

So our theorem follows for univalent B.E. function. The result is generalized to the general class by the principle of subordination [2, pp. 424–425], [9].

REMARK 1. The result is sharp for the B.E. function $F(z)=z^n$, $n=1, 2, \cdots$ and also for Jenkin's functions [4]

$$F(z) = \frac{(1-r^2)^{1/2}z}{1+irz}, \qquad 0 \leqq r < 1. \tag{11}$$

REMARK 2. Jenkin's result [4]

$$F(z) \leqq |z|/(1-|z|^2)^{1/2} \tag{12}$$

follows easily from Theorem 1.

THEOREM 2. *Let* $F(z)=b_1z+b_2z^2+\cdots$ *be a univalent B.E. function. Then we have*

$$|b_n| < e^{-c/2}(n-1)^{-1/2}, \qquad n = 2, 3, \cdots \tag{13}$$

where c is Euler constant.

PROOF. By Lemma 2 and (8), (9) we have

$$\frac{|b_n|^2}{|F'(0)|^2} \leqq \exp\left(\sum_{k=1}^{n-1} k\,|\gamma_{0k}|^2 - \sum_{k=1}^{n-1} 1/k\right) \leqq \frac{\exp\left(-\sum_{k=1}^{n-1} 1/k\right)}{|F'(0)|^2} \tag{14}$$

$$n = 2, 3, \cdots.$$

So $|b_n|^2<e^{-c}(n-1)^{-1}$ which is another form of our theorem. For Jenkin's functions (11) we have $|b_n|^2=(1-r^2)r^{2(n-1)}$. If $1-r^2=1/(n-1)$ we have

$$|b_n|^2 = \frac{1}{n-1}\left(1-\frac{1}{n-1}\right)^{n-1} \sim \frac{1}{e(n-1)}.$$

So the order of magnitude is the best possible and the argument for the constant also follows.

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TECHNION ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL