PUTNAM'S INEQUALITY FOR THE ANNULUS

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1. Putnam [2, p. 52] has conjectured that if T is a seminormal operator on a Hilbert space H then $\pi\|T^*T-TT^*\| \leqq \text{meas}_2\sigma(T)$, where $\text{meas}_2\sigma(T)$ denotes the Lebesgue measure of the spectrum of T. This inequality has been verified by Putnam in the case where the spectrum is convex but holes in the spectrum have caused difficulty. This note proves and utilizes an interesting inequality relating the norm of the self-commutators of the seminormal operators T and T^{-1} to obtain the result $\pi\|T^*T-TT^*\| \leqq \text{meas}_2\sigma(T)$ when $\sigma(T)$ is an annulus.

2. Throughout T will denote a seminormal operator acting on H. As a technical convenience it is assumed that T is hyponormal, that is, the self-commutator $T^*T-TT^*=D$ is positive semidefinite. The following lemma is a standard result on hyponormal operators.

LEMMA 1. *If T is an invertible hyponormal operator then,*

$$\inf_{x\in H}\|T^{-1}x\|/\|x\| = \inf_{x\in H}\|(T^{-1})^*x\|/\|x\| = \text{dist}(0,\sigma(T^{-1})) = \|T\|^{-1}. \tag{2.1}$$

The proof of the following lemma relating the norm of the self-commutator of a hyponormal operator T and the self-commutator of T^{-1} is based on an analysis of Stampfli's proof (see Stampfli [3, p. 469]) that the inverse of a hyponormal operator is hyponormal. The lemma appears in the author's doctoral dissertation.

LEMMA 2. *If T is an invertible hyponormal operator with self-commutator $D=T^*T-TT^*$ $(\geqq 0)$ and $D'=(T^{-1})^*T^{-1}-T^{-1}(T^{-1})^*(\geqq 0)$ then*

$$\frac{\|T\|^{-4}\|D\|}{1+\|D\|\,\|T\|^{-2}} \leqq \|D'\| \leqq \frac{\|T^{-1}\|^{4}\|D\|}{1+\|D\|\,\|T^{-1}\|^{2}}. \tag{2.2}$$

PROOF. Since $T^*T-TT^*=D\geqq 0$, then $T^{-1}T^*T(T^*)^{-1}-I = T^{-1}D(T^*)^{-1}\geqq 0$. Let $T^{-1}D(T^*)^{-1}=B$ and $T^{-1}T^*T(T^*)^{-1}=A$. Then $A-I=B\geqq 0$ and so

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$$I - A^{-1} = C \geqq 0. \tag{2.3}$$

Moreover, $A^{-1} = T^*T^{-1}(T^*)^{-1}T$ so that

$$I - T^*T^{-1}(T^*)^{-1}T = C. \tag{2.4}$$

Then it follows from (2.3) that $D' = (T^*)^{-1}T^{-1} - T^{-1}(T^*)^{-1} = (T^*)^{-1}CT^{-1}$ and, therefore, using Lemma 1

$$\begin{aligned} \|D'\| &= \sup_{\|x\|=1} (D'x, x) \geqq \left(\inf_{\|x\|=1} \|T^{-1}x\|^2 \right) \sup_{\|x\|=1} \frac{(CT^{-1}x, T^{-1}x)}{\|T^{-1}x\|^2} \\ &= \left(\inf_{\|x\|=1} \|T^{-1}x\|^2 \right) \|C\| = \|T\|^{-2}\|C\|. \end{aligned} \tag{2.5}$$

It follows that

$$\|C\|\,\|T\|^{-2} \leqq \|D'\| \leqq \|C\|\,\|T^{-1}\|^2. \tag{2.6}$$

Similarily one obtains

$$\|D\|\,\|T\|^{-2} \leqq \|B\| \leqq \|T^{-1}\|^2\|D\|. \tag{2.7}$$

Since $\|A\| = 1 + \|B\|$ it follows from (2.3) that

$$\|C\| = \frac{\|B\|}{1 + \|B\|}\,. \tag{2.8}$$

Then from equations (2.6) and (2.8) one obtains

$$\frac{\|B\|\,\|T\|^{-2}}{1 + \|B\|} \leqq \|D'\| \leqq \frac{\|B\|\,\|T^{-1}\|^2}{1 + \|B\|}\,. \tag{2.9}$$

Combining equations (2.7) and (2.9) the inequality (2.2) follows easily.

3. The following lemma gives an important inequality on the norm of the self-commutator of an invertible hyponormal operator.

LEMMA 3. *If T is an invertible hyponormal operator and $D = T^*T - TT^*$, then*

$$\|D\| \leqq \|T\|^2 - \|T^{-1}\|^{-2}. \tag{3.1}$$

PROOF. Writing the right inequality in (2.2) for T and T^{-1} one obtains respectively

$$\|D'\| \leqq \frac{\|T^{-1}\|^4\|D\|}{1 + \|T^{-1}\|^2\|D\|} \tag{3.2}$$

and

$$\|D\| \leqq \frac{\|T\|^4\|D'\|}{1 + \|T\|^2\|D'\|}. \tag{3.3}$$

Substituting the upper bound of $\|D'\|$ from equation (3.2) into the inequality (3.3) equation (3.1) is obtained after algebraic simplication.

Since $\|T^{-1}\|^{-1} = \text{dist}\ (0, \sigma(T))$ (Lemma 1) the following corollary results:

COROLLARY. *Let T be hyponormal, if the spectrum of T is the annulus* $A = \{z: 0 < \alpha \leqq |z| \leqq \beta\}$ then $\pi\|T^*T - TT^*\| \leqq \text{meas}_2\sigma(T)$.

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