PRIMES AND ANNIHILATORS

BY STEPHEN MCADAM

Communicated by I. N. Herstein, August 4, 1969

It is well known that in a commutative Noetherian ring, a prime ideal minimal over the annihilator of a finitely generated module is the annihilator of some element of that module. A short proof is given here. We use a lemma which strengthens an observation of Herstein by restricting attention to annihilators within a fixed prime ideal.

LEMMA. *Let R be a commutative ring, P a prime ideal, and A an R-module. Suppose $a \in A$ with $I = \mathrm{Ann}(a) \subset P$. If I is maximal with respect to being an annihilator of an element of A and being contained in P, then I is prime.*

PROOF. Suppose $xy \in I$, but $x \notin I$ and $y \notin I$. Then I is properly contained in (I, y) and $(I, y) \subset \mathrm{Ann}(xa)$. The maximality of I implies $\mathrm{Ann}(xa) \not\subset P$. Pick $s \notin P$, $s \in \mathrm{Ann}(xa)$. We have $I \subset \mathrm{Ann}(sa)$ and $x \in \mathrm{Ann}(sa)$ so that $(I, x) \subset \mathrm{Ann}(sa)$. But I is properly contained in (I, x). Again by the maximality of I, $\mathrm{Ann}(sa) \not\subset P$. Pick $t \notin P$, $t \in \mathrm{Ann}(sa)$. Thus $tsa = 0$ and $ts \in \mathrm{Ann}(a) \subset P$. This is impossible since $s \notin P$ and $t \notin P$.

THEOREM. *Let R be a commutative Noetherian ring, and A a finitely generated R-module. If P is a prime ideal of R minimal over* $\mathrm{Ann}(A)$, *then there is an $a \in A$ with $P = \mathrm{Ann}(a)$.*

PROOF. Say A is generated by $a_1, \cdots, a_n$. Then $\mathrm{Ann}(A)$ equals the intersection, for $i = 1, \cdots, n$, of $\mathrm{Ann}(a_i)$. Since $\mathrm{Ann}(A) \subset P$, we have for some i, $\mathrm{Ann}(a_i) \subset P$. Because R is Noetherian and P contains the annihilator of some element of A, we can find an ideal $I \subset P$ such that I is maximal with respect to being an annihilator of an element of A and being contained in P. Since I is the annihilator of some element of A, we have $\mathrm{Ann}(A) \subset I$. Thus $\mathrm{Ann}(A) \subset I \subset P$. By the lemma, I is prime. Since P is minimal over $\mathrm{Ann}(A)$, we must have $P = I$. Therefore P is the annihilator of some element of A.

The reader will note that we did not use the full strength of R being Noetherian, but only that it satisfied the ascending chain condition on annihilators of elements of A.

AMS Subject Classifications. Primary 1645; Secondary 1610.
Key Words and Phrases. Primes minimal over an annihilator, Noetherian ring.

UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637