ORDER ALGEBRAS

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A partially ordered set P in which every pair of elements has a greatest lower bound is a semigroup, with $pq = p \wedge q$, and hence is naturally associated with a semigroup algebra $Z[P]$ over the integers. For finite P Solomon has given [3] a marvelously ingenious construction of an analogous sort of algebra even when P is not a semilattice and so cannot be made into a semigroup. Semigroup algebras and Solomon's "Möbius algebras" have applications in combinatorial problems involving the underlying orders.

Now in a recent study [2] of valuations and Euler characteristics on lattices Rota introduced an ostensibly quite different sort of algebra he called a "valuation ring" which, rather surprisingly, plays a role like that of a semigroup algebra. More surprising, in view of their entirely different genesis and description, is that Rota's valuation ring can be shown to include Solomon's Möbius algebra as a special case.

Rota's construction, when used to associate such an algebra to a partial order P (which is only one outgrowth of his inquiry), leads in stages through several different structures. The results implicitly provide a recursive procedure for computing products in the valuation ring $V(P)$, but give no direct formula. Solomon, on the other hand, defined his Möbius algebra by giving an explicit, if rather complicated, formula to express products of elements of P as linear combinations of P-elements. The purpose of this note is to determine from Rota's construction an explicit formula for products in $V(P)$ which depends only on the order structure of P. This will show at once that Rota's construction includes Solomon's, and it can be recast in a particularly simple form that clarifies further consequences and applications.

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1. The Rota construction. Let $L=\{S, T, \cdots\}$ be any distributive lattice under $\cup$ and $\cap$, made into a semigroup by setting $ST=S\cap T$. In the semigroup algebra $K[L]$ over a commutative ring K the submodule Q generated by all $S+T-ST-S\cup T$ with S and T in L is an ideal. Since valuations on L are just those functionals which are identically zero on Q, Rota calls the quotient $K[L]/Q$ the *valuation ring* $V(L, K)$. The special case of interest in this note has L the lattice of "order ideals" of a partial order $(P, \leqq)$ and $K=Z$, the ring of integers.

Let P be such that every *cone* $C_p=\{q\in P: q\leqq p\}$ is finite and define L to be the ring of sets generated by all cones, with $\varnothing$ added. Then L is a distributive lattice whose finite elements admit the convenient height function, *ht* $S=|S|$ (number of elements in S). The quotient $Z[L]/Q$ may in this case, because of its likeness to the semigroup algebra of a semigroup, be called the *order algebra*, $V(P)$, of P.

Identifying elements of L with their images in V, Rota extends the identity defining Q to give a general inclusion-exclusion formula that expresses any finite union of lattice elements as a linear combination:

$$(*)\qquad S_1\cup\cdots\cup S_r=\sum_{i=1}^{r}S_i-\sum_{i<j}S_iS_j+\sum_{i<j<k}S_iS_jS_k-\cdots.$$

LEMMA 1. *Any S of finite height in L is a well-defined linear combination, $S=\sum_{p\in P}\phi_S(p)C_p$, of cones contained in S: that is $\phi_S(p)=0$ unless $C_p\subset S$.*

The proof is by induction on the height of S. If ht $S=0$ then $S=\varnothing$ and this is $T+T-TT-T\cup T=0$.

Any other element of finite height in L is either a cone or a finite union of join irreducibles (i.e. cones) of finite height. Now assume the lemma for elements of height $<h$ and suppose *ht* $S=h$. If S is not itself a cone it must be an irredundant union, $S=C_{p_1}\cup\cdots\cup C_{p_r}$, of the maximal cones contained in S. By (*), $S=\sum C_{p_i}-\sum C_{p_i}C_{p_j}+\cdots$, where each term $C_{p_1}\cdots C_{p_k}$ on the right has height $<h$ and hence, by induction, is a well-defined linear combination of cones C_q contained in it, and thus in each C_{p_i}. Then S is the well-defined linear combination gotten by adding all such terms, and furthermore each $C_q\subset C_{p_i}\subset S$.

(If S is written as a nonirredundant union of cones, which can only be done by using all the maximal C_{p_i} in S and other cones C_r contained within some of them, it is easy to show that the *added* contribution from the C_r's amounts to zero.)

Thus V essentially consists of all linear combinations of cones and its multiplication can be taken to define a (commutative Z-algebra) product among P-elements, say $\circ$, by the rule: $x \circ y = \sum_{p\in P} \phi_{xy}(p)\cdot p$ if and only if $C_xC_y = \sum_{p\in P} \phi_{xy}(p)C_p$. This way of writing the $V(P)$ product brings out the analogy with semigroup algebras; of course, $V(P)$ is the integral semigroup algebra on P if and only if P is a semilattice.

2. **Explicit formula for the product.** Rota's procedures show how to compute such products by working upward from minimal elements, but provide no direct way to determine C_xC_y. With only these recursive techniques to build on it is natural to seek an explicit formula by repeated use of induction in the identity (*).

Suppose now that the maximal cones in a given $C_xC_y = C_x \cap C_y$ are $C_{p_1}, \cdots, C_{p_r}$. Then the expansion (*) can be rewritten as

$$(**)\qquad \begin{aligned} C_xC_y &= C_{p_1} \cup \cdots \cup C_{p_r} = \sum_{i=1}^{r} C_{p_i} - \sum_{i<j} C_{p_i}C_{p_j} \\ &\quad + \sum_{i<j<k} C_{p_i}C_{p_j}C_{p_k} - \cdots . \end{aligned}$$

Determining any coefficient $\phi_{xy}(p)$ calls for further expanding each term on the right that is not already a cone until ultimately every term is reduced to a linear combination of cones, and then adding over all terms.

In fact, however, it is simpler to determine first the sum of all $\phi_{xy}(q)$ for q in the filter $F_p = \{q \in P : q \geqq p\}$ above p. Suppose $C_{q_1} \cdots C_{q_t}$ is any term sooner or later arising in the expansion of (**), and that its expression as a linear combination of cones is $\sum \pi_r C_r$. Then the sum of all those π_r for which $r \in F_p$ can be described as the "contribution" of the term $C_{q_1} \cdots C_{q_t}$ to the sum $\sigma_{xy}(p) = \sum_{q\in F_p} \phi_{xy}(q)$.

LEMMA 2. *If $C_{q_1}, \cdots, C_{q_t}$ are cones within $C_x \cap C_y$ then:*

(a) *if there is any i with $p \nleqq q_i$ the contribution of $C_{q_1} \cdots C_{q_t}$ to $\sigma_{xy}(p)$ is* 0;

(b) *if $p \leqq q_i$ for each i this contribution is* 1.

PROOF. If again $C_{q_1} \cdots C_{q_t} = \sum \pi_r C_r$ then whenever $\pi_r \neq 0$ for some $r \in F_p$ it must be that $p \leqq r \leqq q_i$ for each i.

The proof of (b) is by induction on h, the maximum of the heights $ht(C_p, C_{q_i})$ from C_p to C_{q_i}. For $h = 0$ the term $C_{q_1} \cdots C_{q_t} = C_p$ does contribute 1 to $\sigma_{xy}(p)$. Assume the lemma true whenever the maximum of these heights is less than h and now suppose that $q \leqq p_i$ for each i and max $ht(C_p, C_{q_i}) = h$. Notice that if $t=1$ the term is just C_{p_1} and hence does contribute 1 to the sum.

Now with $t>1$ and all $ht(C_p, C_{q_i}) \leqq h$ any cone C_{r_k} which is maximal in $C_{q_1} \cap \cdots \cap C_{q_t}$ must have $ht(C_p, C_{r_k}) < h$ so that for

$$C_{q_1} \cdots C_{q_t} = C_{r_1} \cup \cdots \cup C_{r_s} = \sum_{i=1}^{s} C_{r_i} - \sum_{i<j} C_{r_i} C_{r_j} + \cdots$$

each term on the right contributes 1 to $\sigma_{xy}(p)$, by induction, and hence the total contribution to the sum from $C_{q_1} \cdots C_{q_t}$ is just

$$\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1}\binom{t}{t} = 1.$$

THEOREM. *For any x, y and each $p \in C_x \cap C_y$ the sum $\sigma_{xy}(p) = 1$.*

PROOF. Suppose $C_{p_1}, \cdots, C_{p_r}$ are the maximal cones in $C_x \cup C_y$ with subscripts so chosen that the first s generators $p_1, \cdots, p_s$ are in the filter F_p and the rest are not. The terms $C_{p_{i_1}} \cdots C_{p_{i_k}}$ of the expansion (**) can be split into two classes according as all $p_{i_j} \in F_p$ or not. Then (**) gives $C_x C_y = \sum' + \sum''$ where each term in the former sum ($\sum'$) has all $p_{i_j} \in F_p$ and each term in the latter has at least one $p_{i_j} \ngeqq p$. Now Lemma 2 shows

(a) that the whole contribution to $\sigma_{xy}(p)$ comes from the first sum ($\sum'$) and

(b) that each term in this sum contributes 1. But $\sum'$ is precisely the same as the expansion by (*) of $C_{p_1} \cup \cdots \cup C_{p_s}$ and hence

$$\sigma_{xy}(p) = \binom{s}{1} - \binom{s}{2} + \cdots + (-1)^{s-1}\binom{s}{s} = 1.$$

A straightforward Möbius inversion using the μ-function of P (see [1]) now yields a simple formula for $\phi_{yx}(p)$.

COROLLARY. *For each x, y and p in P: $\phi_{xy}(p) = \sum_{q \in P} \mu(p, q)\sigma_{xy}(q)$. Hence the product, $\circ$, defined by cone multiplication is given by $x \circ y = \sum_{p \in P} \left(\sum_{q \in C_x \cap C_y} \mu(p, q)\right) \cdot p$.*

The product takes this form since $\sigma_{xy}(q) = 1$ or 0 according as $q \in C_x \cap C_y$ or not.

When the order on $P = \{x_0, x_1, x_2, \cdots\}$ can be extended to that of the natural numbers its incidence algebra $\mathfrak{A}(P)$ (see [1]) can be taken to be upper triangular matrices including the Möbius function M with $m_{ij} = \mu(x_i, x_j)$ and its inverse the zeta function $Z(z_{ij} = 1$ or 0 as $x_i \leqq x_j$ or not). Representing each $x_i \in P$ by the column vector with

ith component 1 and all others 0 makes $V(P)$ a left $\mathfrak{A}(P)$ = module consisting of finitely nonzero vectors $x = \sum_i \xi_i x_i$ and having a convolution, $*$, given by $(\sum_i \xi_i x_i) * (\sum_j \eta_j x_j) = \sum_i \sum_j \xi_i \eta_j (x_i \circ x_j)$.

COROLLARY. *If $\cdot$ denotes componentwise multiplication of column vectors, then the product of P-elements is given by $x_i \circ x_j = M(Zx_i \cdot Zx_j)$ and hence the convolution $x * y = M(Zx \cdot Zy)$.*

Thus the operator Z defines a convolution transform $Z(x * y) = Zx \cdot Zy$, and this extends to order algebras the interesting concepts and applications introduced by Tainiter [4] for finite semigroups.

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