HOMOLOGICAL PROPERTIES OF THE RING OF DIFFERENTIAL POLYNOMIALS[1]

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The ring of differential polynomials over a universal differential field (Kolchin [7]), and the ring of twisted polynomials $\overline{F}_2[t, \rho]$, where $\overline{F}_2$ is an algebraic closure of $Z/2Z$ and ρ is the automorphism of $\overline{F}_2$ defined by: $z \mapsto z^2$, "localized" at the multiplicative subset $\{t^k \mid k \text{ an integer} \geqq 0\}$, provide examples of a principal right and left ideal domain R, not a field, that is a right V-ring (i.e., each simple right R-module is injective). Such a ring was conjectured to exist by Carl Faith. Both examples are shown to have a unique simple right R-module. If R is either example, then by definition of a right V-ring, every right R-module has a maximal submodule. Bass proved that if a ring A satisfies the d.c.c. on principal left ideals, then A has a bounded number of orthogonal idempotents and every right A-module has a maximal submodule. The above examples show that the converse is false, thus answering a question raised by Bass [1, p. 470].

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1. **Differential polynomials and right V-rings.**

Throughout this paper each ring R will be a ring with an identity element 1, and each right R-module M will be unitary in the sense that $x1 = x$ for all $x \in M$. Mod-R will denote the category of all right R-modules.

DEFINITION 1. A ring R is a *right V-ring* (after Villamayor) in case the following equivalent conditions are satisfied:

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(1) Each simple right R-module is injective.
(2) Each right ideal is the intersection of maximal right ideals.
(3) Rad $M=0$ for all $M\in$Mod-R.

1.1. PROPOSITION (FAITH). *If R is a prime, right noetherian, right V-ring, then R is simple.*

For a proof of Proposition 1.1 and its generalizations see Faith [**3**, p. 130]. Further results on the structure of right V-rings may also be found in Faith [**4**].

Let k be a field with derivation D and let $k[y, D]$ denote the ring of differential polynomials in the indeterminate y with coefficients in k, i.e., the additive group of $k[y, D]$ is the additive group of the ring of polynomials in the indeterminate y with coefficients in k, and multiplication in $k[y, D]$ is defined by: $ya=ay+D(a)$ for all $a\in k$, and its consequences. Let $f=\sum_{i=1}^{n} a_iy^i\in k[y, D]$, $a_n\neq 0$. Define the degree of f, $\delta(f)=n$. The following properties are immediate:

(1) $\delta(fg)=\delta(f)+\delta(g)$, for all $f, g\in k[y, D]$.

(2) For $f, g\in k[y, D]$, there exist $h, r\in k[y, D]$ such that $f=gh+r$ where $r=0$ or $\delta(r)<\delta(g)$ (a similar algorithm holds on the left).

Thus, by (2), $k[y, D]$ is a principal right and left ideal domain.

Let k be a field of characteristic 0 and D be a derivation of k. A result due to Kolchin asserts the existence of a field $U\supseteq k$ and a derivation $\overline{D}$ of U extending D such that the equation

$$p(x, \overline{D}(x), \cdots, \overline{D}^{(n)}(x))=0, \qquad n \text{ arbitrary},$$

has a solution $\xi\in U$ for all $p(X)\in U[X_1, \cdots, X_{n+1}]-U$. Furthermore, every homogeneous linear differential equation in $\overline{D}$ over U has a nontrivial solution in U. Such a field U is called a ***universal extension of k*** or a ***universal differential field*** (Kolchin [**7**]).

Let k be a universal differential field with derivation D. For the remainder of this section, we shall always denote $k[y, D]$ by R.

1.2. LEMMA. *Given $f=\sum_{i=1}^{n} a_iy^i\in R$, $a_n=1$, there exist $\alpha_i\in k$, $1\leqq i\leqq n$, such that $f=\prod_{i=1}^{n}(y-\alpha_i)$.*

PROOF. By induction on $\delta(f)$. We shall determine $\alpha, b_i\in k$, $2\leqq i\leqq n$, such that

$$f=(y^{n-1}+b_2y^{n-2}+\cdots+b_n)(y-\alpha). \tag{1}$$

By expanding equation (1), equating coefficients, and eliminating the b_i, an equation of the form

$$p(x, D(x), \cdots, D^{(n)}(x))=0 \tag{2}$$

results. By hypothesis, there exists an $\alpha \in k$ satisfying (2).

1.2 implies, in particular, that the only irreducible elements of R are those of degree 1. Hence, $V_\alpha = R/(y-\alpha)R$ is a simple right R-module for all $\alpha \in k$ and conversely.

1.3. LEMMA. *$V_\alpha = R/(y-\alpha)R$ is a divisible right R-module for all $\alpha \in k$.*

PROOF. By 1.2, it suffices to show that

$$V_\alpha(y + \beta) = V_\alpha \qquad \text{for all} \qquad \alpha, \beta \in k.$$

Equivalently, given $h \in R$, $\delta(h) = 0$, there exist f, $g \in R$ such that

$$f(y + \beta) + (y + \alpha)g = h. \tag{1}$$

We shall determine a, $b \in k$ such that

$$a(y + \beta) + (y + \alpha)b = h. \tag{2}$$

Equation (2) is equivalent to an equation of the form

$$D(b) + (\alpha - \beta)b = h. \tag{3}$$

By hypothesis, there exists a $b \in k$ satisfying (3).

1.4 THEOREM. *The ring R has the following properties:*

(a) *R is a principal right and left ideal domain.*

(b) *R is simple.*

(c) *R is a right V-ring.*

(d) *R is not a field.*

(e) *R has, up to isomorphism, a unique simple right R-module.*

PROOF. (a) is obvious by properties (1) and (2) of the ring $k[y, D]$.

(b) is implied by 1.1 since R is a domain.

(c) 1.3, together with the fact that divisibility is equivalent to injectivity in a principal right ideal domain (Faith [4] or Cartan-Eilenberg [2, p. 134]), implies that each simple right R-module is injective.

(d) Obvious.

(e) To show that $R/(y-\alpha)R \cong R/(y-\beta)R$ where α, $\beta \in k$, we observe (see the proof of 1.3) that there exist nonzero a, $b \in k$ such that $a(y-\alpha) = (y-\beta)b$. The map

$$R/(y - \alpha)R \to R/(y - \beta)R$$

defined by

$$r + (y - \alpha)R \mapsto ar + (y - \beta)R$$

is the desired isomorphism.

2. **Twisted polynomials.** Let $\bar{F}_2$ denote an algebraic closure of $Z/2Z$ and ρ, the automorphism of $\bar{F}_2$ defined by: $z \mapsto z^2$. $\bar{F}_2[t, \rho]$ will denote the ring of twisted polynomials in t over $\bar{F}_2$ i.e., the additive group of $\bar{F}_2[t, \rho]$ is the additive group of the ring of polynomials in the indeterminate t with coefficients in $\bar{F}_2$, and multiplication in $\bar{F}_2[t, \rho]$ is defined by: $ta = \rho(a)t$ for all $a \in \bar{F}_2$, and its consequences. It is well known that $\bar{F}_2[t, \rho]$ is a principal right and left ideal domain. Furthermore, it is easy to show that the only two-sided ideals of $\bar{F}_2[t, \rho]$ are those of the form $t^k\bar{F}_2[t, \rho]$, k an integer $\geqq 0$ (Jacobson [5]).

Let $R = \bar{F}_2[t, \rho]$, $M = \{t^k \mid k \text{ an integer} \geqq 0\}$, and $R_M = \{a/m \mid a \in R, m \in M\}$. For a/t^k, $b/t^{k+i} \in R_M$, $a/t^k = b/t^{k+i}$ if and only if $b = t^i a$. We define addition and multiplication in R_M as follows: $a/t^k + b/t^k = (a+b)/t^k$ and $a/t^i b/t^j = \rho^j(a)b/t^{i+j}$ where $\rho^j(a)$ is that element of R obtained by applying ρ^j to all the coefficients of a (Jacobson [6, p. 211]). Clearly, R_M is a simple integral domain, not a field. Moreover, R_M is a principal right and left ideal domain.

2.1. Lemma. *Given* $r = \sum_{i=1}^{n} a_i t^i \in R$, $a_n = 1$, *there exist* $\alpha_i \in \bar{F}_2$, $1 \leqq i \leqq n$, *such that* $r = \prod_{i=1}^{n} (t - \alpha_i)$. *In particular, the irreducible elements of* R *are those of the form* $t - \alpha$, $\alpha \in \bar{F}_2$.

Proof. Analogous to 1.2.

One readily sees that the simple right R_M-modules are of the form R_M/pR_M where $p = t - \alpha$, $\alpha \neq 0 \in \bar{F}_2$.

2.2. Lemma. $R_M/(t-\alpha)R_M$ *is a divisible right* R_M*-module for all* $\alpha \in \bar{F}_2 - \{0\}$.

Proof. Analogous to 1.3.

2.3. Theorem. *The ring* R_M *has the following properties:*

(a) R_M *is a principal right and left ideal domain.*

(b) R_M *is simple.*

(c) R_M *is a right* V*-ring.*

(d) R_M *is not a field.*

(e) R_M *has, up to isomorphism, a unique simple right* R_M*-module.*

Proof. Analogous to 1.4.

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