COERCIVENESS OF THE NORMAL BOUNDARY PROBLEMS FOR AN ELLIPTIC OPERATOR

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Let Ω be a bounded open subset of $\boldsymbol{R}^n$, with smooth boundary Γ (the theory is easily extended to compact manifolds). Let A be a differential operator of order $2m$ $(m \geqq 1)$, with coefficients in $C^\infty(\bar{\Omega})$, such that A is uniformly strongly elliptic and formally selfadjoint in $\bar{\Omega}$. We consider the $L^2(\Omega)$-realizations of A, determined by boundary conditions of the form

$$(1) \qquad \gamma_j u - \sum_{k\in K, k<j} F_{jk}\gamma_k u = 0, \qquad j \in J;$$

here J and K are complementing subsets, each consisting of m elements, of the set $M = \{0, \cdots, 2m-1\}$; the F_{jk} denote (pseudo-) differential operators in Γ of orders $j-k$; and the γ_k denote the standard boundary operators: $\gamma_0 u = u|_\Gamma$, $\gamma_k u = D_n^k u|_\Gamma$, for $u \in C^\infty(\bar{\Omega})$, where $iD_n = \partial/\partial n$ is the interior normal derivative at Γ. (1) is a reduced form of the usual *normal* type of boundary conditions, generalized to include pseudo-differential operators in Γ.

Let $\tilde{A}$ be the operator in $L^2(\Omega)$ defined by

$$(2) \qquad \begin{aligned} D(\tilde{A}) &= \{u \in L^2(\Omega) \mid Au \in L^2(\Omega), u \text{ satisfies } (1)\}, \\ \tilde{A}u &= Au \text{ on } D(\tilde{A}). \end{aligned}$$

(The definition is given a sense by the general concept of boundary value introduced by Lions-Magenes [7]). We shall give below a necessary and sufficient condition on the operators F_{jk} (together with A) in order that $\tilde{A}$ be m-coercive, i.e. satisfies

$$(3) \qquad \operatorname{Re}(\tilde{A}u, u) + \lambda\|u\|_0^2 \geqq c\|u\|_m^2, \qquad \forall u \in D(\tilde{A}),^1$$

for some $c > 0$, $\lambda \in \boldsymbol{R}$. The condition has two parts:

1° it is necessary that the F_{jk} with j and $k \geqq m$ are certain functions of the F_{jk} with j and $k < m$ in order that $\tilde{A}$ be even lower bounded (Theorem 1);

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[1] Here $\|u\|_s$ denotes the norm in the Sobolev space $H^s(\Omega)$, $s \in \boldsymbol{R}$.

2° when 1° is fulfilled, the m-coerciveness is equivalent with an algebraic condition on the *principal symbols* (Theorem 2).
Theorems 1–2 arise as corollaries of a general result (Theorem 3), which permits application of [4], [5].

In [1], Agmon gave an algebraic condition for m-coerciveness of *selfadjoint* realizations defined by *differential* boundary operators; restricted to such realizations, our condition is equivalent with his. Our result also extends those of Fujiwara-Shimakura [3] and Grubb [5], treating certain nonselfadjoint classes of (1). The theory avoids the classical considerations of integro-differential forms, which are not very convenient for the question of necessity. However, our m-coercive $\tilde{A}$ are *variational* in the sense of [5] (i.e., $\tilde{A}+\lambda$ is regularly accretive in Kato's sense, for suitable $\lambda \in \mathbf{R}$).

1. **A necessary condition for lower boundedness.** For a $(p\times q)$

$$E = ((E_{jk}))_{\substack{j=0,\cdots,p-1,\\ k=0,\cdots,q-1}}$$

and two ordered subsets N_1 and N_2 of $\{0,\cdots,p-1\}$ resp. $\{0,\cdots,q-1\}$, we denote the minor $((E_{jk}))_{j\in N_1,k\in N_2}$ by $E_{N_1N_2}$. Similarly for a row- or column-vector $\phi=\{\phi_0,\cdots,\phi_{p-1}\}$ we denote $\{\phi_j\}_{j\in N_1}$ by ϕ_{N_1}. We also use ϕ_{N_1} to indicate a vector $\{\phi_j\}_{j\in N_1}$ indexed by N_1.

Let J, K and M be as above, then we introduce the ordered subsets of M: $M_0=\{0,\cdots,m-1\}$, $M_1=\{m,\cdots,2m-1\}$, $J_0=J\cap M_0$, $J_1=J\cap M_1$, $K_0=K\cap M_0$ and $K_1=K\cap M_1$. When $N\subset M$ we set $N'=\{n \mid 2m-1-n\in N\}$, considered again as an ordered subset of M.

The "Cauchy" boundary operator $\{\gamma_0,\cdots,\gamma_{2m-1}\}$ will be denoted by ρ.

With this notation, (1) is equivalent with

$$\rho_{J_0}u = F_0\rho_{K_0}u, \qquad \rho_{J_1}u = F_1\rho_{K_0}u + F_2\rho_{K_1}u, \tag{4}$$

where F_0, F_1 and F_2 are the matrices of (pseudo-)differential operators (where we put $F_{jk}=0$ for $j\leq k$): $F_0=((F_{jk}))_{j\in J_0,k\in K_0}$, $F_1=((F_{jk}))_{j\in J_1,k\in K_0}$ and $F_2=((F_{jk}))_{j\in J_1,k\in K_1}$. (Evident modifications when empty index sets occur.) They are of *types* $(-k,-j)_{j\in J_0,k\in K_0}$, $(-k,-j)_{j\in J_1,k\in K_0}$ and $(-k,-j)_{j\in J_1,k\in K_1}$, respectively. (The notion of *type* is a convenient generalization of *order* to matrices, the principal symbol σ^0 is defined accordingly, see Hörmander [6], or [5].) Note the way in which F_0 and F_2 are minors of matrices with zeroes in and above the diagonal; we shall say that they are *subtriangular*.

The operator F_0, which maps $\prod_{k\in K_0} H^{s-k}(\Gamma)$ into $\prod_{k\in J_0} H^{s-k}(\Gamma)$, all $s \in \boldsymbol{R}$, can be supplemented with the identity on $\prod_{k\in K_0} H^{s-k}(\Gamma)$ to yield an operator Φ from $\prod_{k\in K_0} H^{s-k}(\Gamma)$ to $\prod_{k\in M_0} H^{s-k}(\Gamma)$:

$$\Phi: \phi_{K_0} \mapsto \psi_{M_0}, \qquad \text{where } \psi_{K_0} = \phi_{K_0}, \quad \psi_{J_0} = F_0\phi_{K_0}.$$

We write in short

$$\Phi = \begin{pmatrix} I_{K_0} \\ F_0 \end{pmatrix}, \qquad \text{where } I_{K_0} = ((\delta_{jk}))_{j,k\in K_0}.$$

The adjoint Φ^* sends ϕ_{M_0} into $\phi_{K_0} + F_0^*\phi_{J_0}$ and is written in short as $\Phi^* = (I_{K_0}\ F_0^*)$. Φ and Φ^* are (pseudo-)differential operators of types $(-k, -j)_{j\in M_0, k\in K_0}$ resp. $(k, j)_{j\in K_0, k\in M_0}$; with an analogous notation for their symbols one has e.g. $\sigma^0(\Phi^*) = (I_{K_0}\ \sigma^0(F_0)^*)$.

At the points of Γ one may write A in normal and tangential coordinates

$$A = \sum_{l=0}^{2m} A_l D_n^l, \tag{6}$$

where the A_l denote differential operators in Γ of orders $2m-l$; A_{2m} is a positive function. Then one has the Green's formula

$$(Au, v) - (u, Av) = \int_\Gamma \mathfrak{A}\rho u \cdot \overline{\rho v}\, d\sigma, \qquad u, v \in C^\infty(\bar{\Omega}), \tag{7}$$

where $\mathfrak{A}$ is a $(2m\times 2m)$-matrix of differential operators in Γ: $\mathfrak{A} = ((\mathfrak{A}_{jk}))_{j,k\in M}$ where each $\mathfrak{A}_{jk}$ has the form iA_{j+k+1}+differential operators of orders *less than* $2m-(j+k+1)$ (we put $A_l = 0$ for $l > 2m$), cf. Seeley [**8**], or [**5**]. We note that $\mathfrak{A}^* = -\mathfrak{A}$, and that $\mathfrak{A}$ is skew-triangular and invertible with $\mathfrak{A}^{-1}$ a differential operator; $\mathfrak{A}$ is elliptic of type $(-k, -2m+j+1)_{j,k\in M}$.

THEOREM 1. *If $\tilde{A}$ is lower bounded, that is, if there exists $\lambda \in \boldsymbol{R}$ such that* $\operatorname{Re}(\tilde{A}u, u) \geq \lambda \|u\|_0^2$, $\forall u \in D(\tilde{A})$, *then $K_0 = J_1'$, and*

$$F_2 = -(\Phi^*\mathfrak{A}_{M_0J_1})^{-1}\Phi^*\mathfrak{A}_{M_0K_1}. \tag{8}$$

(Here $\Phi^*\mathfrak{A}_{M_0J_1}$ is invertible when $K_0 = J_1'$, thanks to the special character of $\mathfrak{A}$ and the subtriangularity of F_0.)

REMARK 1. The case treated by Fujiwara-Shimakura [**3**], Fujiwara [**2**] and Grubb [**5**, **4.3–4.4**] is the case where

$$K_0 = J_1' = \{m-p, \cdots, m-1\}$$

for some $p \leq m$, here F_0 and F_2 are 0 by their subtriangularity; the case in Grubb [**5**, 4.5] takes general K_0 but $F_0 = 0$.

2. **The condition for m-coerciveness.** In accordance with (6), the principal symbol of A may at points $y \in \Gamma$ be written in the form $a(y, \eta, \tau) = \sum_{l=0}^{2m} a^l(y, \eta)\tau^l$, where $a_l(y, \eta)$ denotes the principal symbol of A_l; here η belongs to the fibre at y of the cotangent bundle $T^*(\Gamma)$, and $\tau \in \boldsymbol{R}$. For each (y, η) with $\eta \neq 0$, the polynomial $a(y, \eta, \tau)$ has exactly m roots $\{\tau_i^+(y, \eta)\}_{i=1}^m$ in $\{\lambda \in \boldsymbol{C} \mid \operatorname{Im} \lambda > 0\}$. We can then form the polynomial $\prod_{i=1}^m (\tau - \tau_i^+(y, \eta)) = \sum_{l=0}^m s_l(y, \eta)\tau^l$, and use the coefficients to define the following $(m \times m)$-matrix valued functions on the nonzero subbundle $T^*_\bullet(\Gamma)$ of $T^*(\Gamma)$: $S_0(y, \eta) = ((s_{k-j}(y, \eta)))_{j,k \in M_0}$ and $S_m(y, \eta) = ((s_{m+k-j}(y, \eta)))_{j,k \in M_0}$, where we put $s_l = 0$ for $l \notin [0, m]$. Denoting by $I^\times$ the skew-unit matrix $((\delta_{j,m-1-k}))_{j,k \in M_0}$, we finally introduce $Q = iI^\times \bar{S}_m S_m$, $R = iI^\times \bar{S}_m S_0$, here $\bar{S}$ denotes the complex conjugate of S. (More details in [5, Chapter 4], in fact $Q = A_{2m}^{-1} \sigma^0(\mathfrak{A}_{M_0 M_1})$, and R is the principal symbol of a certain *pseudo-differential* operator in Γ.)

THEOREM 2. *$\tilde{A}$ is m-coercive if and only if it satisfies* (i) *and* (ii):

(i) $K_0 = J_1'$, *and* $F_2 = -(\Phi^* \mathfrak{A}_{M_0 J_1})^{-1} \Phi^* \mathfrak{A}_{M_0 K_1}$.

(ii) *Let* $J_2 = \{j \mid j + m \in J_1\}$, *and let* $E(y, \eta)$ *be the matrix valued function on* $T^*_\bullet(\Gamma)$:

$$E = \sigma^0(\Phi)^* Q_{M_0 J_2} \sigma^0(F_1) + \sigma^0(\Phi)^* R \sigma^0(\Phi), \tag{9}$$

then $E + E^*$ *is positive definite on* $T^*_\bullet(\Gamma)$.

In the affirmative case, $\tilde{A}$ *is* $2m$*-regular* ($\tilde{A}u \in H^s(\Omega) \Rightarrow u \in H^{s+2m}(\Omega)$, $\forall s \geq 0$), *and* $\tilde{A}^*$ *is also* m*-coercive and* $2m$*-regular.*

3. **Explanations and further developments.** The first step in our proof of Theorems 1–2 is the transformation of (4) into an equivalent boundary condition of the form

$$\gamma_{J_0} u = F_0 \gamma_{K_0} u, \qquad \chi_{J_1'} u = G_1 \gamma_{K_0} u + G_2 \chi_{K_1'} u, \tag{10}$$

where γ and χ denote the m-vectors of boundary operators: $\gamma = \rho_{M_0}$, $\chi = \mathfrak{A}_{M_0 M_1} \rho_{M_1} + \frac{1}{2} \mathfrak{A}_{M_0 M_0} \rho_{M_0}$, with which Green's formula (7) takes the simple form: $(Au, v) - (u, Av) = \int_\Gamma (\chi u \cdot \bar{\gamma} \bar{v} - \gamma u \cdot \bar{\chi} \bar{v}) d\sigma$. Note that $\chi = \{\chi_k\}_{k \in M_0}$, where χ_k is of order $2m - k - 1$. There is 1-1 correspondence between the systems (F_0, F_1, F_2) and (F_0, G_1, G_2) (we omit the formulae); G_2^* is again subtriangular.

Assuming, as we may, that the Dirichlet problem for A is uniquely solvable, we define the operator $P_{\gamma,\chi}$ in $\mathfrak{D}'(\Gamma)^m$ by: $P_{\gamma,\chi}\phi = \chi z$, where z is the solution of $Az = 0$ in Ω, $\gamma z = \phi$ (cf. [4], [5]). $P_{\gamma,\chi}$ is a selfadjoint *pseudo-differential operator* in Γ of type $(-k, -2m + j + 1)_{j,k \in M_0}$ (Vainberg-Grušin [9]); its principal symbol is described in detail in [5, Chapter 4].

THEOREM 3. *In addition to the notations introduced above, let* Ψ *be the operator analogous to* Φ *with* F_0 *replaced by* $-G_2^*$. *Let* $X = \Phi(\prod_{k\in k_0} H^{-k-1/2}(\Gamma))$ *and let* $Y = \Psi(\prod_{k\in J_1'} H^{-k-1/2}(\Gamma))$. *Let* Φ_1 *and* Ψ_1 *be the restrictions of* Φ *and* Ψ *with domains* $\prod_{k\in K_0} H^{-k-1/2}(\Gamma)$ *resp.* $\prod_{k\in J_1'} H^{-k-1/2}(\Gamma)$ *and ranges* X *resp.* Y, *clearly they are isomorphisms. Finally, introduce the pseudo-differential operator* $\mathfrak{L}_1$ *of type* $(-k, -2m+j+1)_{j\in J_1' k\in K_0}$:

$$\mathfrak{L}_1 = G_1 - \Psi^* P_{\gamma,\kappa}\Phi.$$

Then $\tilde{A}$ *corresponds, in the sense of* [**4**, *Theorem* III 2.1] (*based on the Dirichlet problem*), *to the operator* $L: X \to Y'$ *defined by*

$$D(L) = \left\{\phi \in X \mid \mathfrak{L}_1 \overset{-1}{\Phi_1} \phi \in \prod_{k\in J_1'} H^{k+1/2}(\Gamma)\right\},$$

$$L\phi = (\overset{*}{\Psi_1})^{-1} \mathfrak{L}_1 \overset{-1}{\Phi_1} \phi, \qquad \textit{when } \phi \in D(L).$$

Theorem 1 follows from this by use of [**4**, Theorem III 4.3]: Lower boundedness of $\tilde{A}$ implies $X \subset Y$, and then by the subtriangularity $\Phi = \Psi$, so that $K_0 = J_1'$ and $F_0 = -G_2^*$, which leads to (8). Note that then $X = Y$.

Theorem 2 uses [**5**, Corollary 2.4]: $\tilde{A}$ is m-coercive if and only if L is m-coercive, i.e., $X \subset Y$ and $\exists c > 0$, $\lambda \in \boldsymbol{R}$ so that

$$\operatorname{Re}\langle L\phi_Y, \phi_{Y'}\rangle + \lambda \|\phi\|^2_{\{-k-1/2\}} \geq c\|\phi\|^2_{\{m-k-1/2\}} \text{ on } D(L).^2$$

This is equivalent with a similar property for $L_1 = \Psi_1^* L \Phi_1$, which is a certain "realization" of $\mathfrak{L}_1$, and here the property amounts (besides $\Phi = \Psi$) to the positive definiteness of $\sigma^0(\mathfrak{L}_1 + \mathfrak{L}_1^*)$ (in fact $E = A_{2m}^{-1}\sigma^0(\mathfrak{L}_1)$); the computations resemble those in [**5**]. The last statement in Theorem 2 uses the *ellipticity* of $\mathfrak{L}_1$.

REMARK 2. The *selfadjoint* m-coercive $\tilde{A}$ are characterized by Theorem 2 (i), (ii), plus selfadjointness of $G_1 = \Phi^* \mathfrak{A}_{M_0 J_1} F_1 + \frac{1}{2}\Phi^* \mathfrak{A}_{M_0 M_0}\Phi$ (then E is also selfadjoint).

REMARK 3. Theorem 3 gives a basis for the discussion of many other properties of $\tilde{A}$, because of the way in which they are preserved by the correspondence between $\tilde{A}$ and L, see [**4**], [**5**]. Regarding coerciveness, we mention that:

1° the conditions in Theorem 2 are also necessary and sufficient for $(m-\epsilon)$-coerciveness with $\epsilon \in [0, 1/2[$ (cf. Fujiwara-Shimakura [**3**]),

2° the discussion of $(m-1/2)$-coerciveness in Fujiwara [**2**] (related to subellipticity [**6**]) seems extendable to the present case,

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[2] $\|\phi\|_{\{s-k-1/2\}}$ denotes the norm in $\Pi_{k\in M_0} H^{s-k-1/2}(\Gamma)$.

3° a necessary condition for lower boundedness ("0-coerciveness") is the positive *semidefiniteness* of $\sigma^0(\mathfrak{L}_1+\mathfrak{L}_1^*)$ (cf. [5, Theorem 4.3]). Let us mention that lower boundedness $+2m$-regularity do *not* imply m-coerciveness as in the selfadjoint case; examples using pseudo-differential operators: take $\mathfrak{L}_1$ elliptic with $\mathfrak{L}_1^*=-\mathfrak{L}_1$.

Concerning extensions of the results to operators A that are merely strongly elliptic, let us mention that the case $K_0=J_1'=\{m-p, \cdots, m-1\}$ has been treated by Fujiwara [2]; the device of [2] does not extend to our general case.

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