MATCHING THEOREMS FOR COMBINATORIAL GEOMETRIES

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1. **Introduction.** Let $G(S)$ and $G(T)$ be combinatorial geometries of finite rank on sets S and T, respectively, and let $R \subseteq S \times T$ be a binary relation between the points of $G(S)$ and $G(T)$. By a *matching* from $G(S)$ into $G(T)$, we understand a one-one function f from an independent set $A \subseteq S$ onto an independent set $B \subseteq T$ with $(a, f(a)) \in R$ for all $a \in A$. In this note, we present a characterization of matchings of maximum cardinality, a max-min theorem, and a number of related results. In the case when $G(S)$ and $G(T)$ are both free geometries, Theorems 1 and 2 reduce to "the Hungarian method" as introduced by Egerváry and Kuhn [**1**], and to the König-Egerváry theorem, respectively. Corollary 2 for the case when $G(S)$ is a free geometry and $G(T)$ arbitrary was first discovered by Rado [**6**] (see also Crapo-Rota [**2**]). When both $G(S)$ and $G(T)$ are free geometries, Corollary 2 reduces to the well-known SDR theorem.

2. **Terminology.** For an arbitrary geometry $G(S)$, the closure operator will be denoted by J and the rank function by r. $(G(S), G(T), R)$ shall denote the system of the two geometries together with R, and $R(S') = \{y \mid \text{ there is some } x \in S' \text{ with } (x, y) \in R\}$ for $S' \subseteq S$. Let (A, B, f) denote a matching from A onto B. $M = \{(a, f(a)), a \in A\}$ is called the *edge set* of the matching (A, B, f), and we adopt the convention $M = (A, B, f)$. The common cardinality of A, B, M is called the *size* $\nu(M)$ of the matching. A *support* of $(G(S), G(T), R)$ is a pair (C, D) of closed sets, where $C \subseteq S$, $D \subseteq T$, such that $(c, d) \in R$ implies at least one of $c \in C$, $d \in D$ holds. The *order* λ of a support (C, D) is defined as $\lambda(C, D) = r(C) + r(D)$. Finally, an *augmenting* chain with

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respect to the matching $M=(A, B, f)$ is a sequence of $2n+1$ distinct pairs $(a_0', b_1'), (b_1, a_1), (a_1', b_2'), \cdots, (b_n, a_n), (a_n', b_{n+1}')$ such that

$$(1) \qquad (a_i,b_i) \in M, \qquad (a_i', b_{i+1}') \in R - M,$$

$$(2) \qquad a_0' \in S - J(A), \qquad b_{n+1}' \in T - J(B),$$

$$(3) \qquad \begin{aligned} a_i' &\in J(A), \qquad & a_i' &\notin J\left(\left(A - \bigcup_{j=1}^{i} a_j\right) \cup \bigcup_{j=1}^{i-1} a_j'\right), \\ b_i' &\in J(B), \qquad & b_i' &\notin J\left(\left(B - \bigcup_{j=1}^{i} b_j\right) \cup \bigcup_{j=1}^{i-1} b_j'\right) \end{aligned}$$

for $1 \leqq i \leqq n$.

3. The main results.

THEOREM 1. *A matching $M=(A, B, f)$ in $(G(S), G(T), R)$ is of maximum size if and only if there does not exist an augmenting chain with respect to M.*

THEOREM 2. $\max_{M \text{ matching}} \nu(M) = \min_{(C,D) \text{ support}} \lambda(C, D)$.

BRIEF OUTLINE OF PROOF OF THEOREMS 1 AND 2. First, it is easily seen that by means of an augmenting chain we can increase a given matching M, since by conditions (2) and (3) the sets

$$A' = \left(A - \bigcup_{j=1}^{n} a_j\right) \cup \bigcup_{j=0}^{n} a_j',$$

$$B' = \left(B - \bigcup_{j=1}^{n} b_j\right) \cup \bigcup_{j=1}^{n+1} b_j'$$

are independent. Further, we clearly have $\nu(M) \leqq \lambda(C, D)$ for any matching M and any support (C, D).

Assume now there is no augmenting chain with respect to (A, B, f). Put $C_0 = S - J(A)$, then $R(C_0) \subseteq J(B)$. Let B_1 be the minimal subset of B such that $R(C_0) \subseteq J(B_1)$, $A_1 = f^{-1}(B_1)$ and $C_1 = S - J(A - A_1)$. In general, having constructed C_{i-1}, we define B_i as the minimal subset of B such that $R(C_{i-1}) \cap J(B) \subseteq J(B_i)$, and set $A_i = f^{-1}(B_i)$ and $C_i = S - J(A - A_i)$. This way we construct three monotonically increasing sequences of sets A_i, B_i, C_i and since all the B_i's are contained in B, these sequences must terminate after a finite number of, say, m steps.

The crucial part of the argument consists in showing that $R(C_n) \subseteq J(B)$ for all $n = 0, \cdots, m$. This is accomplished by disproving the opposite through construction of an augmenting chain with respect to M. Now since $R(C_m) \subseteq J(B_m)$, i.e., $R(S-J(A-A_m)) \subseteq J(B_m)$, we infer that $(J(A-A_m), J(B_m))$ constitutes a support with order equal to the size of M. Thus M is a matching of maximum cardinality and the equality in Theorem 2 holds.

COROLLARY 1. *For $A \subseteq S$, define the deficiency of A as $\delta_S(A) = r(S) - r(S-A) - r(R(A))$, and let $\delta_S = \max_{A \subseteq S} \delta_S(A)$.* Then

$$\max_{M \text{ matching}} \nu(M) = \min_{(C,D) \text{ support}} \lambda(C, D) = r(S) - \delta_S.$$

We have

$$\begin{aligned} r(S) - \delta_S &= r(S) - \max_{A \subseteq S} (r(S) - (r(S - A) - r(R(A))) \\ &= \min_{A \subseteq S} (r(S - A) + r(R(A))) = \min_{A \subseteq S} (r(A) + r(R(S - A))), \end{aligned}$$

and the minimum is clearly obtained by some closed set A. But then $(A, J(R(S-A)))$ is a support for $(G(S), G(T), R)$ and the conclusion follows.

COROLLARY 2 (GENERALIZED MARRIAGE THEOREM). *Given $(G(S), G(T), R)$, then* $\max_{M \text{ matching}} \nu(M) = r(S)$ *if and only if $r(S) - r(S-A) \leqq r(R(A))$ for all $A \subseteq S$.*

COROLLARY 3. *Let (A, B, f) be a matching in $(G(S), G(T), R)$ and suppose it is not of maximum size, then there exists a matching $(A' \cup a, B' \cup b, f')$ such that $J(A') = J(A)$, $J(B') = J(B)$, and $a \notin J(A')$, $b \notin J(B')$.*

This follows immediately from the definition of augmenting chains, part (3).

COROLLARY 4 (*See also* [2], [3], [4]). *Given $(G(S), G(T), R)$, where $G(S)$ is a free geometry. Define a new independence structure on S by calling $A \subseteq S$ independent if and only if there exists a matching (A, B, f) for some B and f. This defines a pregeometry on S, called the transversal pregeometry with respect to $(G(S), G(T), R)$.*

Corollary 3 applied to $(G(S'), G(T), R \cap (S' \times T))$ for $S' \subseteq S$ shows that every independent subset $A \subseteq S'$ as defined above can be embedded in one of maximum (and by Corollary 1, constant) size.

It should be remarked that Corollary 4 ceases to be true for arbitrary geometries $G(S)$. The function r^* given by the definition of independent sets in Corollary 4 and by the formula in Corollary 1 as $r^*(S') = r(S') - \delta_{S'}$ for $S' \subseteq S$ is unit-increasing, but fails to be semimodular in general. For the same reason one cannot prove Theorem

2 along the lines suggested by Ore [**5**] although this approach works when $G(S)$ is a free geometry.

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