A NILPOTENT LIE ALGEBRA WITH NILPOTENT AUTOMORPHISM GROUP

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Recent work of Stein, Knapp, Koranyi and others has been concerned with nilpotent Lie groups which admit expanding automorphisms, that is, semisimple automorphisms whose eigenvalues are all greater than one in absolute value (cf. [1], [3], [5]). It was an open question whether all nilpotent Lie groups admit expansions.

We first present a result due to Louis Auslander (oral communication) which establishes the existence of a class of Lie groups which do admit expanding automorphisms (§1). We then present an example of a nine dimensional Lie group which does not have an expanding automorphism (§2). In our work it is more convenient to use Lie algebra language than Lie group language. As is well known, for connected simply connected nilpotent Lie groups the choice of group or algebra language is a matter of taste.

I wish to take this opportunity to thank Louis Auslander for suggesting this problem to me.

1. Let us begin by recalling some definitions. Details may be found in [2, Chapter V] or in [6, Chapter 5]. Henceforth algebra will mean algebra over the reals.

We say that $\mathfrak{L}$ is a *free Lie of algebra rank r* if there exist r elements $X_1, \cdots, X_r \in \mathfrak{L}$ which generate $\mathfrak{L}$ qua algebra and which enjoy the following universal mapping property: any function from the set $\{X_1, \cdots, X_r\}$ to any algebra $\mathcal{L}$ extends to a unique algebra homomorphism $\mathfrak{L} \rightarrow \mathcal{L}$.

Define the ideals $\mathfrak{L}^i$, $i=1, 2, \cdots$ of $\mathfrak{L}$ as follows:

$$\mathfrak{L}^1 = \mathfrak{L}, \qquad \mathfrak{L}^{i+1} = [\mathfrak{L}^i, \mathfrak{L}].$$

An ideal $\mathcal{I}$ of $\mathfrak{L}$ is *homogeneous* if the vector space $\mathcal{I}$ is isomorphic to the direct sum of $\mathcal{I} \cap \mathfrak{L}^i / \mathcal{I} \cap \mathfrak{L}^{i+1}$, $i=1, 2, \cdots$. We shall say that the Lie algebra $\mathcal{L}$ is homogeneous if $\mathcal{L}$ is isomorphic to $\mathfrak{L}/\mathcal{I}$ with $\mathfrak{L}$ free and $\mathcal{I}$ homogeneous.

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THEOREM. *Any homogeneous Lie algebra admits expanding automorphisms.*

PROOF. We shall construct an expanding automorphism $\alpha\colon \mathfrak{L}\to\mathfrak{L}$ and show $\alpha(\mathfrak{I})\subset\mathfrak{I}$. Let λ be any real number of absolute value greater than one. Let $X_1, \cdots, X_r$ be free generators for $\mathfrak{L}$. It follows that the vector space spanned by the X_i is of dimension r and is a compliment of $\mathfrak{L}^2$. Set $\alpha(X_i)=\lambda X_i$, and extend α to an automorphism of $\mathfrak{L}$ by the universal mapping property.

There is a basis of $\mathfrak{L}$ consisting of monomials in the X_i, that is, bracket products of the X_i. A monomial μ is of degree d if $\mu\in\mathfrak{L}^d-\mathfrak{L}^{d+1}$; equivalently, μ is nonzero and is a product involving precisely d of the X_i, counting multiplicity. As α is a homomorphism, $\alpha(\mu)=\lambda^d\mu$. Hence α acts as scalar multiplication by λ^d on the vector space $\mathcal{H}^d$ spanned by the monomials of degree d. Now $\mathcal{H}^d$ is isomorphic to $\mathfrak{L}^d/\mathfrak{L}^{d+1}$ under the map which sends each monomial to its coset modulo $\mathfrak{L}^{d+1}$, and we may decompose $\mathcal{H}^d=L^d\oplus H^d$, where $H^d=\mathcal{H}^d\cap\mathfrak{I}$. Since $\mathfrak{I}$ is homogeneous, it is the direct sum of its subspaces H^d, and therefore $\alpha(\mathfrak{I})\subseteq\mathfrak{I}$. Thus α induces an automorphism $\beta\colon \mathcal{L}\to\mathcal{L}$. But $\mathcal{L}=\mathfrak{L}/\mathfrak{I}$ is isomorphic to the direct sum of $\mathcal{H}^d/\mathcal{H}^d\cap\mathfrak{I}\cong L^d$, so β restricted to L^d is scalar multiplication by λ^d. Thus all eigenvalues of β are greater than one in absolute value, or, β is expanding.

2. We recall that a nilpotent Lie algebra of class c is an algebra for which $\mathcal{L}^c\neq 0$, $\mathcal{L}^{c+1}=0$ where the $\mathcal{L}^i$ are defined as above. Nilpotent algebras are algebra generated by any elements which generate $\mathcal{L}/\mathcal{L}^2$. A free nilpotent algebra of rank r and class c may be viewed as the quotient $\mathfrak{L}/\mathfrak{L}^{c+1}$, where $\mathfrak{L}$ is free of rank r. We note that any elements which form a basis of $\mathfrak{L}/\mathfrak{L}^2$, when viewed as elements of $\mathfrak{L}/\mathfrak{L}^{c+1}$ enjoy the universal mapping property for maps into nilpotent algebras of class at most c.

We now denote by $\mathfrak{L}$ the free nilpotent Lie algebra of rank 2 and class 6. We select a monomial basis $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{U}_8, \boldsymbol{U}_9$ of $\mathfrak{L}$ such that

$\boldsymbol{X}_1, \boldsymbol{X}_2$	is a basis of	$\mathfrak{L}$ mod $\mathfrak{L}^2$,
$\boldsymbol{V}$	is a basis of	$\mathfrak{L}^2$ mod $\mathfrak{L}^3$,
$\boldsymbol{W}_1, \boldsymbol{W}_2$	is a basis of	$\mathfrak{L}^3$ mod $\mathfrak{L}^4$,
$\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3$	is a basis of	$\mathfrak{L}^4$ mod $\mathfrak{L}^5$,
$\boldsymbol{Z}_1, \cdots, \boldsymbol{Z}_6$	is a basis of	$\mathfrak{L}^5$ mod $\mathfrak{L}^6$,
$\boldsymbol{U}_1, \cdots, \boldsymbol{U}_9$	is a basis of	$\mathfrak{L}^6$.

$\mathfrak{L}$	X_1	X_2	V	W_1
X_2	$-V$			
V	W_1	W_2		
W_1	Y_3	Y_2	Z_5-Z_4	
W_2	Y_2	Y_1	Z_3-Z_2	$2U_5-U_4-U_6$
Y_1	Z_2	Z_1	U_3-U_2	
Y_2	Z_4	Z_3	U_5-U_4	
Y_3	Z_6	Z_5	U_8-U_7	
Z_1	$2U_2-U_3$	U_1		
Z_2	$3U_4-3U_5+U_6$	U_2		
Z_3	U_4	U_3		
Z_4	U_7	U_5		
Z_5	U_8	U_6		
Z_6	U_9	$2U_8-U_7$		

FIGURE 1

With respect to this basis, the multiplication table of $\mathfrak{L}$ is given in Figure 1. Unlisted products are obtained by antisymmetry or are zero; the entry in row A column B is the product $[AB]$.

Let $\mathfrak{I}$ be the ideal of $\mathfrak{L}$ generated by the set

$$S = \{Y_1 - Y_3, Z_4, Z_1 - U_4, Z_2 - 3Z_3 - U_4\}.$$

To obtain a vector space basis for $\mathfrak{I}$, we use the fact that $\mathfrak{I}$ is generated by the union of the sets S_i, $i=1, 2, 3$ where

$$S_1 = S, \qquad S_{i+1} = \{[WX_j]: j = 1, 2; W \in S_i\}.$$

We remark that $S_4=0$ since $[\mathfrak{L}^i, \mathfrak{L}^j] \subset \mathfrak{L}^{i+j}$ and $S \subset \mathfrak{L}^4$. Thus $\mathfrak{I}$ has basis:

$$Y_1 - Y_3,\ Z_1, - U_4,\ Z_2 - 3Z_3 - U_4,\ Z_2 - Z_6,\ Z_4, Z_5 - U_4,$$
$$U_1,\ U_2,\ U_3,\ U_5,\ U_6,\ U_7,\ U_8,\ U_9 - 3U_4.$$

Set $\mathcal{L}=\boldsymbol{\mathcal{L}}/\mathcal{I}$, and let π: $\boldsymbol{\mathcal{L}}\to\mathcal{L}$ be the canonical projection. We have that $\mathcal{L}$ is 9-dimensional and has basis

$$X_1 = \pi\boldsymbol{X}_1, \quad X_2 = \pi\boldsymbol{X}_2, \quad V = \pi\boldsymbol{V}, \quad W_1 = \pi\boldsymbol{W}_1, \quad W_2 = \pi\boldsymbol{W}_2,$$
$$Y_1 = \pi\boldsymbol{Y}_1, \quad Y_2 = \pi\boldsymbol{Y}_2, \quad Z = \pi\boldsymbol{Z}_3, \quad U = \pi\boldsymbol{U}_4.$$

We shall now show that each automorphism of $\mathcal{L}$ is unipotent. It then follows from a theorem of E. Kolchin [**4**] that the group of automorphisms is nilpotent.

Let us first observe that if the automorphism $\alpha:\mathcal{L}\to\mathcal{L}$ induces the identity automorphism of $\mathcal{L}/\mathcal{L}^2$, it follows that α induces identity automorphisms of $\mathcal{L}^i/\mathcal{L}^{i+1}$ for each $i=1, \cdots, 6$ and therefore that α is unipotent. We must establish that $\alpha(X_i)=X_i \bmod \mathcal{L}^2$, $i=1, 2$. However it does not suffice to consider $\alpha(X_i) \bmod \mathcal{L}^2$ for the ideal $\mathcal{I}$ is not homogeneous.

We consider therefore

$$\alpha(X_1) = a_1X_1 + a_2X_2 + a_3V \mod \mathcal{L}^3,$$
$$\alpha(X_2) = b_1X_1 + b_2X_2 + b_3V \mod \mathcal{L}^3.$$

We shall use the notation

$$\lambda_{ij} = a_ib_j - a_jb_i, \qquad i, j = 1, 2, 3$$

where $\lambda=\lambda_{12}\neq 0$ as α induces an automorphism of $\mathcal{L} \bmod \mathcal{L}^2$.

One computes that

$$\begin{aligned}\alpha(V) &= [\alpha X_1, \alpha X_2] = \lambda V + \lambda_{31}W_1 + \lambda_{32}W_2 \mod \mathcal{L}^4\\ \alpha(W_1) &= [\alpha V, \alpha X_1] = a_1\lambda W_1 + a_2\lambda W_2 + (a_1\lambda_{31} + a_2\lambda_{32}) Y_1\\ &\quad + (a_2\lambda_{31} + a_1\lambda_{32}) Y_2 \mod \mathcal{L}^5\\ \alpha(W_2) &= [\alpha V, \alpha X_2] = b_1\lambda W_1 + b_2\lambda W_2 + (b_1\lambda_{31} + b_2\lambda_{32}) Y_1\\ &\quad + (b_2\lambda_{31} + b_1\lambda_{32}) Y_2 \mod \mathcal{L}^5\end{aligned}$$

which implies therefore that

$$\alpha(U) = [\alpha W_1, \alpha W_2] = \lambda^3 U = [\alpha W_1, \alpha V] = -2a_2\lambda^2 Z \mod \mathcal{L}^6 \tag{1}$$

and so we have

$$a_2 = 0, \qquad \lambda = a_1b_2. \tag{2}$$

Simplifying our computations accordingly we now find

$$\begin{aligned}\alpha(Y_1) &= [\alpha W_1, \alpha X_1] = a_1^2\lambda Y_1 + 3a_1^2\lambda_{31}Z \mod \mathcal{L}^6\\ &= [\alpha W_2, \alpha X_2] = (b_1^2 + b_2^2)\lambda Y_1 + 2b_1b_2\lambda Y_2 \mod \mathcal{L}^5.\end{aligned} \tag{3}$$

Thus $b_1b_2=0$, $\lambda=a_1b_2\neq 0$ so

$$(4) \qquad b_1 = 0, \qquad a_1^2 = b_2^2.$$

Now use (3) to obtain

$$\alpha(U) = [\alpha Y_1, \alpha X_2] = a_1^2 b_2 \lambda U = a_1\lambda^2 U$$

and (1) then implies that $a_1\lambda^2=\lambda^3$ or

$$(5) \qquad b_2 = 1, \qquad \lambda = a_1.$$

Lastly, $[Y_1, X_1]=3Z+U=3[Y_2, X_2]+U$; and equations (1) and (3) yield

$$[\alpha Y_1, \alpha X_1] = 3\lambda^2 Z + \lambda^2 U, \qquad 3[\alpha Y_2, \alpha X_2] + \alpha U = 3\lambda^2 Z + \lambda^3 U$$

hence $\lambda^2=\lambda^3$ and so

$$(6) \qquad \lambda = a_1 = 1.$$

Equations (2), (4), (5) and (6) then state that α does induce the identity on $\mathfrak{L}/\mathfrak{L}^2$.

We remark that the construction of $\mathfrak{L}$ and the computations of this section are valid over any field whose characteristic is not 2 or 3.

ADDED IN PROOF. It has been brought to the attention of the author that the nilpotent Lie algebra constructed by J. Dixmier and W. G. Lister (*Derivations of nilpotent Lie algebras*, Proc. Amer. Math. Soc. **8** (1957), 155–158) has no expanding automorphisms. This algebra is shown to have a niplotent derivation algerba, which implies that the eigenvalues of any automorphism are roots of unity. The automorphism group of this algebra is however not nilpotent.

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