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HOLOMORPHIC MAPPINGS INTO TIGHT MANIFOLDS

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This paper gives an extension (*Proposition* 3) of *Theorem* C of H. Wu's paper [4], as well as a few other results. The terminology will be that of [4].

If M and N are complex manifolds $A(M, N)$ will denote the set of holomorphic mappings between M and N. It is a topological space under the topology of uniform convergence on compact subsets of M. If f_i is a sequence in $A(M, N)$ and g is in $A(M, N)$ then $f_i \to g$ will mean that the f_i's converge to g in this topology. A pair (N, d), where N is a complex manifold and d is a distance on N, will be called *tight* iff $A(M, N)$ is equicontinuous with respect to d for all complex manifolds M. In fact (N, d) is tight iff $A(B^n, N)$ is equicontinuous with respect to d, where B^n here denotes the unit ball in C^n. For details see *Part* I of [4].

Our basic lemma, interesting for its own sake, is

PROPOSITION 1. *Let M be a connected complex manifold, U an open subset of M, and (N, d) be tight. For $f \in A(M, N)$ define $i_U(f) \in A(U, N)$ to be the restriction of f to U. Then i_U is a homeomorphism of $A(M, N)$ into $A(U, N)$.*

PROOF. i_U is one-to-one because U is open in M. If $f_i \to g$ in $A(M, N)$ it is clear that $i_U(f_i) \to i_U(g)$. Thus i_U is continuous, and it remains only to show that $i_U(f_i) \to i_U(g)$ in $A(U, N)$ implies that $f_i \to g$ in $A(M, N)$.

Suppose $i_U(f_i) \to i_U(g)$ in $A(U, N)$. Let $\mathfrak{U} = \{ V \subset M$: V open in M and $i_V(f_i) \to i_V(g)$ in $A(V, N)\}$. Partially order $\mathfrak{U}$ by inclusion. If $V_1 \subset V_2 \subset V_3 \subset \cdots$ is a totally ordered chain in $\mathfrak{U}$, it is clear that $V = \bigcup V_j$ is a member of $\mathfrak{U}$. Since $U \in \mathfrak{U}$, $\mathfrak{U}$ is not empty, so Zorn's Lemma implies that $\mathfrak{U}$ contains maximal elements. Let U_0 be one such. We will show that $U_0 = M$.

If not, $\partial U_0 = \overline{U}_0 - U_0$ is not empty. Let $x \in \partial U_0$ and $\epsilon > 0$. Since N

is tight there is a neighborhood, V, of x such that $y \in V$ implies $d(h(x), h(y)) < \epsilon/3$ for all holomorphic mappings $h: M \to N$. Pick such a $y \in V \cap U_0$ and pick i_0 such that $i > i_0$ implies $d(f_i(y), g(y)) < \epsilon/3$. Then $i > i_0$ implies that

$$d(f_i(x), g(x)) \leqq d(f_i(x), f_i(y)) + d(f_i(y), g(y)) + d(g(y), g(x)) < \epsilon.$$

This shows that $f_i(x) \to g(x)$.

Let B be a *taut* (see [4, p. 199]) neighborhood of $g(x)$ in N. Since M is tight and $f_i(x) \to g(x)$ there is a connected neighborhood, W, of x in M such that $f_i(W) \subset B$ for large i. Now the set of holomorphic mappings from W to B, $A(W, B)$ is a *normal family* [4, p. 197]. If $\{i(j)\}$ is any subsequence if Z^+ then $f_{i(j)}(x) \to g(x)$ and it follows that there is a subsequence $\{j(s)\}$ of Z^+ such that $i_W(f_{i(j(s))}) \to h$, where h is a member of $A(W, B)$. But $W \cap U_0$ is open, so the h must coincide with g on $W \cap U_0$, and hence $h = i_W(g)$. Thus $i_W(f_i) \to i_W(g)$ and $i_{W \cup U_0}(f_i) \to i_{W \cup U_0}(g)$, so U_0 is not maximal, a contradiction. Hence $U_0 = M$ and $f_i \to g$ in $A(M, N)$. Q.E.D.

Proposition 1 is not true for general complex manifolds. Let $M = N = C$, the complex plane, and $U = B^1$, the open unit disk. For n a positive integer, define $f_n(z) = (1 - 1/n)z + (1/n)z^n$. f_n approaches the identity uniformly on compact subsets of B^1, but $f_n(z) \to \infty$ for $|z| > 1$.

COROLLARY 2. *Let (M, d) be a tight manifold and $U \subset M$ be open. Suppose $f: M \to M$ is holomorphic and for some subsequence, $\{i(s)\}$ of Z^+, $i_U(f^{i(s)}) \to \mathrm{id}_U$. Then f is an automorphism of M.*

PROOF. By *Proposition* 1, $f^{i(s)} \to \mathrm{id}$ on M. This gives the conclusion by repeating verbatim the argument at the end of the proof of *Theorem* C [4, p. 208]. Q.E.D.

PROPOSITION 3. *Let M be a tight manifold with respect to some distance d, $p \in M$, and $f: M \to M$ holomorphic with $f(p) = p$. Then*

(i) $|\det df_p| \leqq 1$,

(ii) *df_p is the identity matrix iff f is the identity on M,*

(iii) *$|\det df_p| = 1$ iff f is an automorphism of M.*

PROOF. Let W be a *taut* [4, p. 199] neighborhood of p which is contained in some coordinate neighborhood of p. By equicontinuity there is a neighborhood, U, of p in M such that $g(U) \subset W$ for any holomorphic mapping $g: M \to M$. We may suppose in fact that U is an open ball in the coordinates about p. Now (i), (ii) and the $\Leftarrow$ part of (iii) are proved exactly as in the proof of *Theorem* C [4, pp. 205, 206].

For the remainder of (iii) it follows as on p. 207 of [4] that there is a subsequence, $\{i(s)\}$ of Z^+ such that $i_U(f^{i(s)}) \to \mathrm{id}_U$. Now *Corollary* 2 shows f is an automorphism of M. Q.E.D.

PROPOSITION 4. *Let M be a tight complex manifold, U be open and relatively compact in M, f: $M \to M$ holomorphic, and $i_U(f)$ an automorphism of U. Then f is an automorphism of M.*

PROOF. M is tight iff M is hyperbolic, in the sense of Kobayashi [3, p. 465]. This is shown in [1, *Part* II: 3.8]. For $r > 0$, let

$$U_r = \{x \in U : \kappa(x, \partial U) > r\}.$$

(Here κ is the Kobayashi distance on M.) If $x \in U_r$, $x = f(y)$ for some $y \in U$, and $\kappa(y, U) \geq \kappa(x, f(\partial U)) \geq \kappa(x, \partial U) > r$. Thus $U_r \subset f(U_r)$, and by [1, *Part* III: 1.5], $f(U_r) = U_r$ and $i_{U_r}(f)$ is an automorphism of U_r.

We can choose a subsequence of Z^+, $\{i(m)\}$, and points y_m in U_r such that $\kappa(f^{i(m)}(y_m), y_m) < 1/m$ for each m. Since $\overline{U}_r$ is compact in U we may pass to a subsequence and assume $y_m \to p$, where p is some point in $\overline{U}_r$.

Now for $x \in U_r$, x is in U_s for some $s > 0$, and by the argument in the first paragraph of this proof, $f^i(x) \in U_s$ for all positive integers i. Since $\overline{U}_s$ is compact in U, $\{f^{i(m)}(x)\}$ is relatively compact in U. Since $f^{i(m)}$ is an equicontinuous family with respect to κ, it follows from the Ascoli Theorem that there is a subsequence of $\{i(m)\}$, which we shall again denote by $\{i(m)\}$, such that $i_U(f^{i(m)}) \to g$ in U, where g is a holomorphic mapping from U to itself, and $g(p) = p$. Since $i_{U_r}(f^i)$ is an automorphism of U_r for each $r > 0$ and each positive integer i, it follows from the relative compactness of U_r in U that $g(U_r) = U_r$ for each $r > 0$, and [1, *Part* III: 1.5] shows that g is an automorphism of U_r for each $r > 0$. Hence g is an automorphism of U.

By *Proposition* 3, $|\det dg_p| = 1$. From this and the argument on p. 207 of [4] it follows that for some subsequence, $\{k(s)\}$, of $Z^+ g^{k(s)} \to \mathrm{id}_U$. It is now easily seen that for some subsequence of Z^+, $\{i(s)\}$, $i_U(f^{i(s)}) \to \mathrm{id}_U$, and *Corollary* 2 shows that f is an automorphism of M. Q.E.D.

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