CONTRACTIVE PROJECTIONS AND PREDICTION OPERATORS[1]

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Communicated by Jack Schwartz, July 7, 1969

1. **Introduction.** The purpose of this note is to present some results on characterizations of subspaces of a general class of Banach function spaces (BFS) admitting contractive projections onto them, and to include an application to nonlinear prediction (and approximation) theory.

Let L^ρ be the subspace of all measurable scalar functions f on (Ω, Σ, μ) with $\rho(f) = \rho(|f|) < \infty$, where $\rho(\cdot)$ is a function norm, i.e., a norm with the additional properties

(i) $0 \leqq f_n \uparrow \Rightarrow \rho(f_n) \uparrow$, and

(ii) $\rho(\cdot)$ verifies the triangle inequality for infinite sums. Then L^ρ is also complete, called a BFS, (cf. [6] and [4]). It will also be assumed, for convenience, that $0 \leqq f_n \uparrow f \Rightarrow \rho(f_n) \uparrow \rho(f)$, the Fatou property. $\rho(\cdot)$ is an absolutely continuous norm (a.c.n.) if for each $f \in L^\rho$, $\rho(f\chi_{A_n}) \not\to 0$ for any A_n in Σ, $A_n \downarrow \emptyset$. If $\mathfrak{X}$ is a B-space, $L^\rho_{\mathfrak{X}}$ is the space of $\mathfrak{X}$-valued strongly measurable functions f on Ω, with $\rho(|f|_{\mathfrak{X}}) < \infty$, where $\rho(\cdot)$ is as above. Then $L^\rho_{\mathfrak{X}}$ is also complete. Finally let $\mathfrak{M}^\rho_{\mathfrak{X}} = \overline{\mathrm{sp}}\{fx : f \in L^\rho,\ x \in \mathfrak{X}\} \subset L^\rho_{\mathfrak{X}}$. A projection is a linear idempotent operator.

The projection problem, stated at the outset, has been first treated for $L^\rho = L^1$ in [5], and a more detailed consideration of the same case, with $\mu(\Omega) < \infty$, has been given in [2]. If $L^\rho = L^p$, also with $\mu(\Omega) < \infty$, it was then considered in [1], and these results were extended for $L^\rho = L^\Phi$, the Orlicz spaces, with a.c.n. and μ σ-finite, in [10]. The general solution of the problem in the scalar case, and a less general one in the vector case, will be given below.

2. **Contractive projections.** Let $\mathcal{S} \subset L^\rho$ be a closed subspace. If $L^\rho \neq L^2$, then, as is well known, not every $\mathcal{S}$ is the range of a bounded projection. The positive solution is given by the following result for L^ρ-spaces. (An operator T is positive if $Tf \geqq 0$ for $f \geqq 0$.)

THEOREM 1. *If (Ω, Σ, μ) is a measure space, let $L^\rho(\Sigma)$ be the BFS defined above. Consider the statements:*

(a) *$\mathcal{S}$ is the range of a (positive) contractive projection in $L^\rho(\Sigma)$.*

(b) *there is an isometric isomorphism $\Psi: L^\rho(\Sigma) \mapsto L^\rho(\Sigma)$, ($\Psi$ = identity) such that*

[1] Supported, in part, under the NSF grant GP-8777.

(i) $\Psi(\mathcal{S})$ *is a B-lattice, i.e., a selfadjoint space with real functions forming a lattice, and*

(ii) $0 \leqq f_n \in \Psi(\mathcal{S}), f_n \uparrow f, f \in L^\rho(\Sigma) \Rightarrow f \in \Psi(\mathcal{S})$.

(c) *there is a (positive) isometric isomorphism between some* $L^\rho(\mathcal{B})$ *on some measure space* $(S, \mathcal{B}, \mu_1)$ *and* $\mathcal{S}$.

(d) *same as* (c) *except "topological equivalence" replaces "isometric isomorphism."*

Then one has (c)$\Rightarrow$(a)$\Leftrightarrow$(b)$\Rightarrow$(d). *In case* $\rho(\cdot)$ *also verifies,* $\chi_A \in L^\rho(\Sigma)$ *for each* $A \in \Sigma$ *with* $\mu(A) < \infty$, *then* (a)$\Leftrightarrow$(c) *also holds.*

REMARK. If $\rho(f) = \int_0^1 |f| \, d\mu/x$, with $\Omega = [0, 1]$, μ = Leb.meas., then $\rho(\cdot)$ is a function norm, but $\rho(\chi_\Omega) = \infty$. Thus the last condition of the theorem is a restriction on ρ. It can be shown easily that b(ii) automatically holds if ρ is an a.c.n., but will be needed otherwise.

This result is proved through several isomorphisms using equivalent measure spaces and the results of [**13**]. However, for an application of the latter, a first reduction is needed and is provided by the following result which has independent interest.

THEOREM 2. *If* $L^\rho(\Sigma)$ *is a BFS on* (Ω, Σ, μ), *then there exists a measure space* $(S, \mathcal{B}, \nu)$ *where* S *is a locally compact space,* $\mathcal{B}$ *is a* σ*-field generated by the compact subsets of* S *and* ν *is a measure assigning finite measure for compacts, in terms of which* $L^\rho(S, \mathcal{B}, \nu)$, *or* $L^\rho(\mathcal{B})$, *is isometrically (and lattice) isomorphic to* $L^\rho(\Sigma)$. *Moreover each* f *in* $L^\rho(\mathcal{B})$ *has a* σ *compact support. If there exists a strictly positive element in* $L^\rho(\Sigma)$, *then* S *can be chosen compact, so that* $(S, \mathcal{B}, \nu)$ *is a finite measure space.*

If μ is σ-finite then a strictly positive element always exists in $L^\rho(\Sigma)$ (e.g., a weak unit, cf. [**6**, p. 153]) and the last part contains this case. This result is proved using a method of proof of ([**8**, Theorem 2.1]) and some results of [**13**]. (See also [**3**] for the L^1-case.) With this reduction, the problem of Theorem 1 can be transfered to $L^\rho(\mathcal{B})$. Then it can be isometrically embedded in $L^\rho(\tilde{\mathcal{B}})$ on a localizable measure space $(\tilde{S}, \tilde{\mathcal{B}}, \tilde{\nu})$ where $\mathcal{B}$ goes, under an algebraic isomorphism, into a subring of $\tilde{\mathcal{B}}$, [**13**, Theorem 3.4]. Then the proof is successively reduced to the case of finite measure space where the methods and ideas of [**2**] and [**10**] can be generalized and used. In this way the full result of Theorem 1 is established.

In general there will be many contractive projections onto $\mathcal{S}$, when one exists. The following gives a uniqueness result.

PROPOSITION 3. *Suppose* $L^\rho(\Sigma)$ *is a rotund (=strictly convex) and smooth (=norm is Gâteaux differentiable) reflexive space on* (Ω, Σ, μ). *Then a closed subspace* $\mathcal{S} \subset L^\rho(\Sigma)$ *can be the range of atmost one contrac-*

tive projection. If in particular $\mathcal{S}=L^\rho(\mathcal{B})$, $\mathcal{B}\subset\Sigma$, *a σ-field, then there exists a unique positive contractive projection onto* $\mathcal{S}$, *namely the (generalized) conditional expectation* $E^{\mathcal{B}}: L^\rho(\Sigma)\mapsto L^\rho(\mathcal{B})$.

The case of $L^\rho=L^p$, $1<p<\infty$, $\mu(\Omega)<\infty$, of the above result was obtained in ([**1**, p. 392]). The general form of P is not-simple. The following case is illustrative.

PROPOSITION 4. *Let* $P: L^\rho(\Sigma)\mapsto L^\rho(\mathcal{B})$ *be a contractive projection (which exists by Theorem 1), where* $\mathcal{B}\subset\Sigma$ *is a σ-field with* $\mu_{\mathcal{B}}$ *σ-finite, and* $L^\rho(\Sigma)$ *is a BFS. Then there exists a locally integrable function g such that*

(i) $P(\cdot)=E^{\mathcal{B}}(g\cdot)$, *and*

(ii) $E^{\mathcal{B}}(g)=1$ *a.e., where* $E^{\mathcal{B}}$ *is the conditional expectation relative to* $\mathcal{B}$.

This shows that while $E^{\mathcal{B}}$ itself is a contractive projection onto $L^\rho(\mathcal{B})$, it is not the general form of the operator. If ρ is an a.c.n., then it can be shown that $g=1$ a.e. here, and this is not necessarily true in the general case. The above two results are proved by an extension of the methods of [**10**]. A special case of the above proposition for L^Φ-spaces, with $\mu(\Omega)<\infty$, was discussed in [**11**].

For the case of $\mathfrak{M}^\rho_{\mathfrak{X}}$ spaces, the following result holds.

THEOREM 5. *Let* $L^\rho(\Sigma)$ *and* $\mathfrak{M}^\rho_{\mathfrak{X}}$ *be as defined in* §1. *If* $\mathcal{S}\subset L^\rho(\Sigma)$ *is a closed subspace, let* $\mathcal{S}_{\mathfrak{X}}=\overline{\mathrm{sp}}\{fx: f\in\mathcal{S},\ x\in\mathfrak{X}\}\subset\mathfrak{M}^\rho_{\mathfrak{X}}$. *Also let* $\chi_A\in L^\rho(\Sigma)$ *for each* $A\in\Sigma$ *with* $\mu(A)<\infty$. *Then the following four statements are equivalent*:

(i) $\exists$ *contractive projection* $P: L^\rho(\Sigma)\mapsto\mathcal{S}$.

(ii) $\exists$ *contractive projection* $P: \mathfrak{M}^\rho_{\mathfrak{X}}\mapsto\mathcal{S}_{\mathfrak{X}}$.

(iii) $\exists L^\rho(\mathcal{B}_1, \mu_1)$, *on some measure space* $(S_1, \mathcal{B}_1, \mu_1)$ *and* $\mathcal{S}$ *is isometrically isomorphic to* $L^\rho(\mathcal{B}_1, \mu_1)$.

(iv) $\mathcal{S}_{\mathfrak{X}}$ *is isometrically isomorphic to* $\mathfrak{M}^\rho_{\mathfrak{X}}(\mathcal{B}_1, \mu_1)$.

This result is proved on using Theorem 1, and the fact that $L^\rho\otimes_\gamma\mathfrak{X}\subset\mathfrak{M}^\rho_{\mathfrak{X}}$ and is dense in the latter (see [**9**]). Here $\otimes_\gamma$ is the greatest cross-norm, and one then uses a result on projections in cross-spaces [**12**, p. 58]. The general case of $L^\rho_{\mathfrak{X}}$ itself does not seem to follow in this way. The above one already includes the $L^p_{\mathfrak{X}}$, $1\leqq p\leqq\infty$ case.

3. **Prediction operators.** A subspace $M\subset L^\rho$ is said to be a *Tshebyshev subspace* if for each $x\in L^\rho$ there is a unique $x_0\in M$ with $\rho(x-x_0)=\min\{\rho(x-y): y\in M\}$. The operator $P_M: x\mapsto x_0\in M$, is

called a *prediction operator* in nonlinear prediction theory. Though $P_M^2 = P_M$, it is not linear in general. If it is linear, the powerful methods of linear analysis will be available in their study. So this is a natural question to treat. If P_M is linear, then $Q = I - P_M$ is a contractive projection with M as its null space (and the converse also holds). This is the connection between projections and predictions, and a solution can be presented as follows.

THEOREM 6. *Let $M \subset L^\rho$ be a Tshebyshev subspace, and P_M be the prediction operator for M. If P_M is linear then the quotient space L^ρ/M is topologically equivalent to $L^\rho(\mathcal{B})$ on some measure space $(S, \mathcal{B}, \mu_1)$. Conversely, if L^ρ/M is isometrically isomorphic to $L^\rho(\mathcal{B})$ on some $(S, \mathcal{B}, \mu_1)$ then P_M is linear.*

In case $\chi_A \in L^\rho$ for each $A \in \Sigma$, $\mu(A) < \infty$, then the above can be stated as: P_M is linear$\Leftrightarrow L^\rho/M$ is isometrically isomorphic to an $L^\rho(\mathcal{B})$. If $L^\rho = L^p$, $1 < p < \infty$, $\mu(\Omega) < \infty$, the latter has been obtained in [1]. The general case can be proved quickly with the results of the preceding section. However, it was noted in [10], that for the case $L^\rho \neq L^2$, M must be relatively complicated since P_M will not be linear if M is of the form $L^\rho(\Sigma_1)$, $\Sigma_1 \subset \Sigma$, a σ-field.

The proofs of all the results above involve first a characterization of the adjoint space $(L^\rho)^*$ of L^ρ. This is involved. It is obtained by generalizing the work of ([7] and [4]) appropriately. With these results (and those of [9]), and of [13], the above bare sketch is completed. The details and related results will be published separately.

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