ISOMETRIC EMBEDDINGS[1]

BY ROBERT E. GREENE

Communicated by Avner Friedman, September 16, 1968

This note states results extending those of Nash [2] on isometric embeddings of Riemannian manifolds in euclidean spaces; proofs and further details will be given elsewhere.

Let M be a d-dimensional C^∞ manifold. For convenience, we assume throughout that manifolds, whether compact or not, are connected. A *metric* on M is defined to be a quadratic form on the tangent bundle of M; *note that there is no assumption of nondegeneracy*. We shall assume that all metrics are C^∞. A *Riemannian metric* on M is a metric whose restriction to the tangent space T_q at a point $q \in M$ is positive definite, for all $q \in M$. A *pseudo-Riemannian*, or *indefinite*, *metric* is a metric whose restriction to the tangent space at each point is nondegenerate; if the nondegenerate restriction to T_q has n negative eigenvalues and p positive eigenvalues, with $p+n=d$, the metric is said to have signature (p, n) at q. The connectedness of M implies that the signature is independent of the choice of $q \in M$.

R^m will denote euclidean m-dimensional space, with the standard flat, positive definite metric, unless otherwise indicated; R_n^p denotes euclidean $(n+p)$-dimensional space with flat metric of signature (p, n). Thus $R_0^m = R^m$. Let F be a C^∞ map, $F: M \to R_n^p$, and let g be a metric on M; F is said to be isometric for g if $F^*(\cdot) = g$ where "$\cdot$" denotes the metric for R_n^p indicated above. Note that if g is Riemannian and F is isometric for g, then F is necessarily an immersion and $n+p \geqq d = \dim M$; for a general metric g, however, F need not be an immersion. We shall concern ourselves with the question: given M and a metric g on M, for what R_n^p do there exist isometric immersions, or isometric embeddings, $F: M \to R_n^p$?

1. A geometric argument for general metrics. Nash [2] guarantees the existence of isometric embeddings in some Riemannian euclidean space for any manifold with a Riemannian metric. The following argument reduces the general metric case to the Riemannian case, but requires higher dimension in the receiving euclidean space than necessary.

---

[1] This research was conducted while the author was supported by a National Science Foundation Graduate Fellowship.

PROPOSITION. *Suppose any Riemannian metric on M has an isometric immersion in R^k. Then any metric g on M has an isometric immersion (embedding) in $R^k_{2d}(R^k_{2d+1})$.*

OUTLINE OF PROOF. Using Whitney's standard results on immersions and embeddings, one obtains a "large" immersion or embedding $E: M \to R^{2d}$ or R^{2d+1} such that $E^*(\cdot)+g$ is Riemannian. Then, if F is isometric for $E^*(\cdot)+g$, $F \times E: M \to R^k_{2d}$ (or R^k_{2d+1}) is isometric for g, where F maps to the k positive-eigenvalue coordinates and E to the $2d$ (or $2d+1$) negative ones.

2. The compact case.

THEOREM 1. *Let M be compact, d-dimensional, with metric g. There is an embedding $F: M \to R^k_k$, $k=d(d+5)/2$, which is isometric for g.*

In [2], Nash shows that given a Riemannian metric g on a compact manifold M, there is an isometric embedding of M in euclidean $d(3d+11)/2$-dimensional space, where the euclidean space is flat Riemannian; the above theorem reduces the dimension required for the euclidean space and extends the result to arbitrary g, but the euclidean space is pseudo-Riemannian, even if g is Riemannian.

3. The noncompact case. The following Theorem 2 derives the existence of isometric immersions for noncompact manifolds from the isometric immersions of compact ones; the lemma, applicable to compact or noncompact manifolds, then provides isometric embeddings in the noncompact case. More specific results are given in Theorem 3.

THEOREM 2. *Suppose that, for any metric g on the $2d+1$-dimensional sphere S^{2d+1}, there is an isometric immersion of S^{2d+1} in R^n, where R^n has a flat Riemannian or pseudo-Riemannian metric. Then, if M is any d-dimensional manifold with metric, there is an isometric immersion of M in R^{2n}, where R^{2n} has a flat Riemannian or pseudo-Riemannian metric. Furthermore, if, for any Riemannian metric g on S^{2d+1}, there is an isometric immersion in R^p_n, then any manifold of dimension d with a Riemannian metric has an isometric immersion in R^{2p}_{2n}.*

LEMMA. *If a d-dimensional manifold M, compact or not, has an isometric immersion in R^p_n for any metric g on M, then M has an isometric embedding in R^{p+2d}_n for any g. Further, if, for every Riemannian metric g, there is an isometric immersion in R^p, then there is an isometric embedding in R^{p+2d+1}.*

THEOREM 3. *Every manifold M of dimension d, compact or not, has some isometric embedding in R_1^k, $k=(2d+1)(2d+6)$, for every metric g on M. If g is a Riemannian metric, M has an isometric embedding in R^k, $k=(2d+1)(6d+14)$.*

Theorem 3 improves Nash's result that a noncompact d-dimensional manifold M with Riemannian metric has an isometric embedding in R^k, $k=d(d+1)(3d+11)/2$.

4. **The local case.** If a metric g is defined on an open set U in a manifold M, then, given an open set V with $\overline{V}\subset U$, there is a metric $\bar{g}$ defined on M such that $\bar{g}|V=g|V$. Thus the global statements above imply corresponding local conclusions. However, in the local case, using A. Friedman's results [1] on the local analytic case, we can considerably improve the dimensional requirements for the cases of Riemannian and pseudo-Riemannian metrics.

THEOREM 4. *Let g be a metric defined on an open set in R^d and u a point of U. Suppose that g has signature (p_1, n_1) at u, $p_1+n_1=d$. Then there is an open set V, with $u\in V$, such that there is an embedding of V in R_n^p, $n+p=d(d+3)/2$, which is isometric for $g|V$. If g is Riemannian at u, then n may be taken equal to 0. More generally, n and p may be chosen subject only to the restrictions $n\geqq n_1$, $p\geqq p_1$, $n+p=d(d+3)/2$.*

REFERENCES

1. A. Friedman, *Local isometric imbedding of Riemannian manifolds with indefinite metrics*, J. Math. Mech. **10** (1961), 625–649.

2. J. Nash, *The isometric imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–64.

UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720