SPECTRAL PROPERTIES OF HIGHER DERIVATIONS ON SYMMETRY CLASSES OF TENSORS

BY MARVIN MARCUS[1]

Communicated by Gian-Carlo Rota, July 14, 1969

Let V be a finite dimensional vector space over a field R, and let $T\colon V\rightarrow V$ be a linear transformation on V. The transformation T may be extended in a unique way to a derivation $\Omega_1(T)$ of the tensor algebra $\otimes V=\sum_{p=0}^{\infty}\otimes_1^p V$ by defining $\Omega_1(T)$ on each of the homogeneous components $\otimes_1^p V$ by

$$\Omega_1(T)v_1\otimes\cdots\otimes v_p=\sum_{j=1}^{p} v_1\otimes\cdots\otimes v_{j-1}\otimes Tv_j\otimes v_{j+1}\otimes\cdots\otimes v_p$$

when $p>1$, and $\Omega_1(T)=0$ if $p=0$, $\Omega_1(T)=T$ if $p=1$ (Greub [2, p. 67]).

In this announcement we outline an extension of this definition to higher order derivations in general symmetry classes of tensors. We also state some of the eigenvalue properties of higher derivations and show how these may be applied to some classical matrix problems. Let H be a subgroup of S_p and let χ be a character of degree 1 on H. (We assume that the order of H exceeds the characteristic of R.) If U is a vector space over R and $\phi(v_1,\cdots,v_p)$ is a p-multilinear function on the Cartesian product $\times_1^p V$ to U, then ϕ is said to be symmetric with respect to H and χ if

$$\phi(v_{\sigma(1)},\cdots,v_{\sigma(p)})=\chi(\sigma)\phi(v_1,\cdots,v_p)$$

for any $\sigma\in H$ and arbitrary vectors $v_i\in V$. A pair (P,μ) consisting of a vector space P over R and a p-multilinear function $\mu:\times_1^p V\rightarrow P$, symmetric with respect to H and χ, is a symmetry class of tensors associated with H and χ if

(i) $\langle \text{rng}\ \mu\rangle=P$ (i.e. the linear closure of the range of μ is P);

(ii) for any vector space U over R and any p-multilinear ϕ symmetric with respect to H and χ, there exists a linear function $h\colon P\rightarrow U$ such that $\phi=h\mu$.

$$\begin{tikzcd} \times_1^p V \arrow[rr, "\mu"] \arrow[dr, "\phi"'] & & P \arrow[dl, "h"] \\ & U & \end{tikzcd}$$

[1] This research was supported by the U. S. Air Force Office of Scientific Research, Applied Mathematics Division under grant AFOSR 698-67.

The symmetry class (P, μ) is unique to within canonical isomorphism. We shall denote the *decomposable* element $\mu(v_1, \cdots, v_p)$ of P by $v_1 * \cdots * v_p$.

For $1 \leqq r \leqq p \leqq n = \dim V$ let $Q_{r,p}$ denote the set of $C_{p,r}$ strictly increasing sequences of length r chosen from $1, \cdots, p$. For any $\omega = (\omega_1, \cdots, \omega_r)$ in $Q_{r,p}$ let $\pi_\omega(T): \otimes_1^p V \rightarrow \otimes_1^p V$ denote the tensor product

$$\pi_\omega(T) = I \otimes \cdots \otimes \overset{\omega^1}{T} \otimes \cdots \otimes \overset{\omega^r}{T} \otimes \cdots \otimes I$$

of the identity and the transformation T in which T occurs in positions $\omega_1, \cdots, \omega_r$. Next let $\delta_r(T)$ be the sum

$$\delta_r(T) = \sum_{\omega \in Q_{r,p}} \pi_\omega(T).$$

THEOREM 1. *Any symmetry class of tensors (P, μ) is an invariant subspace of $\delta_r(T)$ and*

$$\delta_r(T) v_1 * \cdots * v_p = \sum_{\omega \in Q_{r,p}} v_1 * \cdots * Tv_{\omega_1} * \cdots * Tv_{\omega_r} * \cdots * v_p.$$

The restriction of $\delta_r(T)$ to the subspace P will be denoted by $\Omega_r(T)$. The map $\Omega_r(T): P \rightarrow P$ is called the rth derivation on P induced by T. In the event $r = p$ the map $\Omega_p(T): P \rightarrow P$ is usually called the transformation induced by T on the symmetry class P.

THEOREM 2. *If T: $V \rightarrow V$ then*

$$\Omega_p(I + sT) = \sum_{r=0}^{p} s^r \Omega_r(T)$$

where $\Omega_0(T)$ is the identity transformation on the symmetry class P.

It is possible to give a complete eigenvalue analysis of $\Omega_r(T)$. In order to do this we must introduce a minimal amount of combinatorial machinery. Thus let $\Gamma_{p,n}$ denote the totality of sequences of positive integers $\alpha = (\alpha_1, \cdots, \alpha_p)$, $1 \leqq \alpha_i \leqq n$, $i = 1, \cdots, p$. Two sequences $\alpha, \beta \in \Gamma_{p,n}$ are said to be equivalent modulo H, $\alpha \sim \beta$, if there exists a permutation $\sigma \in H$ such that $\alpha^\sigma = (\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(p)}) = \beta$. The equivalence relation $\sim$ partitions $\Gamma_{p,n}$ into equivalence classes. Let Δ denote a system of distinct representatives for $\sim$ chosen so that each sequence in Δ is lowest in lexicographic order in the equivalence class in which it lies. For any $\alpha \in \Delta$ let H_α denote the stabilizer group of α in H, i.e., $\sigma \in H_\alpha$ if $\alpha^\sigma = \alpha$. Let $\bar{\Delta}$ be the subset of Δ consisting of those α for which

$$\sum_{\sigma \in H_\alpha} \chi(\sigma) = \nu(\alpha) \neq 0. \tag{1}$$

Since χ is a character of degree 1 on H, it follows that $\nu(\alpha)$ is, in fact, the order of H_α and that the sum on the left in (1) is either 0 or $\nu(\alpha)$ under any circumstances. It is well known that if $e_1, \cdots, e_n$ is a basis of V then the tensors $e_\alpha^* = e_{\alpha_1} * \cdots * e_{\alpha_p}$, $\alpha \in \bar{\Delta}$, constitute a basis for the symmetry class P. If T: $V \to V$ and $Te_i \in \langle e_1, \cdots, e_i \rangle$, $i = 1, \cdots, n$, then the ordered basis $\{e_1, \cdots, e_n\}$ is said to be a triangular basis for T.

THEOREM 3. *If $\{e_1, \cdots, e_n\}$ is a triangular basis for T then the lexicographically ordered basis $\{e_\alpha^*, \alpha \in \bar{\Delta}\}$ of P is a triangular basis for $\Omega_r(T)$: $P \to P$, $r = 1, 2, \cdots$. Moreover, if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of T then the eigenvalues of $\Omega_r(T)$ are the numbers*

$$E_r(\lambda_{\alpha_1}, \lambda_{\alpha_2}, \cdots, \lambda_{\alpha_p}), \qquad \alpha \in \bar{\Delta},$$

where E_r is the rth elementary symmetric function of the indicated numbers.

If V is a unitary space then an inner product is induced in P which satisfies

$$(u_1 * \cdots * u_p, v_1 * \cdots * v_p) = h^{-1} d((u_i, v_j)) \tag{2}$$

where $((u_i, v_j))$ denotes the p-square matrix whose (i, j) entry is (u_i, v_j),

$$d(X) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{p} x_{i,\sigma(i)}$$

for any p-square matrix X, and h is the order of H.

THEOREM 4. *Let V be a unitary space with inner product (u, v) and let T: $V \to V$. Then the adjoints with respect to the inner products in V and P satisfy*

$$\Omega_r(T^*) = \Omega_r(T)^*.$$

Moreover,

(a) *if T is normal, (hermitian, positive-definite, or positive-semi-definite), so is $\Omega_r(T)$;*

(b) *if $E = \{e_1, \cdots, e_n\}$ is an orthonormal basis of eigenvectors of T then the set of tensors*

$$E^* = \{(h/\nu(\alpha))^{1/2} e_\alpha^*, \ \alpha \in \bar{\Delta}\}$$

is an orthonormal basis of eigenvectors of $\Omega_r(T)$.

COROLLARY. *Assume V is unitary and let P be either the pth exterior product $\wedge^p V$ or the space $V^{(p)}$ of completely symmetric tensors (i.e., $H = S_p$, $\chi = \mathrm{sgn}$, or $\chi \equiv 1$). Let A be the matrix representation of T with respect to the basis E and let $\Omega_r(A)$ be the matrix representation of $\Omega_r(T)$ with respect to the basis E^*. Then if $\alpha \in Q_{p,n}$ the α, α entry of $\Omega_r(A)$ is*

$$\sum_{\omega \in Q_{r,p}} d(A[\alpha_{\omega_1}, \cdots, \alpha_{\omega_r}]). \tag{3}$$

In (3) d is the determinant in case $P = \wedge^p V$ and the permanent in case $P = V^{(p)}$. The matrix $A[\alpha_{\omega_1}, \cdots, \alpha_{\omega_r}]$ is the r-square principal submatrix of A lying in rows $\alpha_{\omega_1}, \cdots, \alpha_{\omega_r}$.

This corollary leads to a generalization of a number of results proved over the last twenty years by several authors.

THEOREM 5. *Let A be an n-square normal matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ and let B be a principal p-square submatrix of A. For $1 \leqq r \leqq p \leqq n$ let $d_r(B)$ denote the sum of all $C_{p,r}$ r-square principal subdeterminants of B and let $\rho_r(B)$ denote the sum of all $C_{p,r}$ r-square principal subpermanents of B. Then*

(a) *$d_r(B)$ is in the convex hull of the $C_{n,p}$ numbers*

$$E_r(\lambda_{\alpha_1}, \cdots, \lambda_{\alpha_p}), \qquad \alpha \in Q_{p,n};$$

(b) *$\rho_r(B)$ is in the convex hull of the $C_{n+p-1,p}$ numbers*

$$E_r(\lambda_{\alpha_1}, \cdots, \lambda_{\alpha_p}),$$

where α runs over all nondecreasing sequences of length p chosen from $1, \cdots, n$.

For $r = 1$ and A hermitian, this result was proved by Fan [1]. For general r, $p = n$, and A positive-definite hermitian, the result was proved by Marcus and McGregor [3]. For A hermitian the result was proved by Marcus, Moyls, and Westwick [4]. Quite recently Thompson proved the result for d_r when A is normal [5].

Theorem 5 is typical of a number of results which can be proved by essentially the same techniques. Some of these applications will appear in the forthcoming Proceedings of the Second Symposium on Inequalities edited by O. Shisha (Academic Press, New York, 1969).

REFERENCES

1. Ky Fan, *On a theorem of Weyl concerning eigenvalues of linear transformations*. I, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 652–655.

2. W. H. Greub, *Multilinear algebra*, Springer-Verlag, New York, 1967.

3. Marvin Marcus and J. L. McGregor, *Extremal properties of hermitian matrices*, Canad. J. Math. **8** (1956), 524–531.

4. Marvin Marcus, B. N. Moyls and R. Westwick, *Extremal properties of hermitian matrices*. II, Canad. J. Math. **11** (1959), 379–382.

5. R. C. Thompson, *A note on normal matrices*, Canad. J. Math. **15** (1963), 220–225.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106