UNIFORM ALGEBRAS ON CURVES

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1. **Results.** A recent result of H. S. Shapiro and A. L. Shields [4] states that if f and g are continuous complex valued functions on the unit interval I such that together they separate the points of I and also that f alone separates all but one pair of points, then the closed subalgebra of $C(I)$ generated by f and g is all of $C(I)$. Two generalizations are:

THEOREM. *Let A be a separating uniform algebra on I such that there exists an f in A which is locally* 1-1, *then $A = C(I)$.*

THEOREM. *Let A be a separating uniform algebra on I generated by two functions f and g such that there is a compact totally disconnected subset E of I such that*

(i) *$f|E$ is constant, and*

(ii) *f separates every pair of points of I not both of which are in E. Then $A = C(I)$.*

The proofs use the notion of analytic structure in a maximal ideal space. J. Wermer first obtained results along these lines and further contributions were made by E. Bishop and H. Royden and then by G. Stolzenberg [5] who proved

STOLZENBERG'S THEOREM. *Let $X \subseteq C^n$ be a polynomially convex set. Let $K \subseteq C^n$ be a finite union of $\mathcal{C}^1$-curves. Then $(X \cup K)\hat{}\ - X \cup K$ is a (possibly empty) pure* 1*-dimensional analytic subset of $C^n - X \cup K$. (See* [5] *for the notation and definitions.)*

A further result of Stolzenberg (and Bishop) is that a $\mathcal{C}^1$ arc $K \subseteq C^n$ is polynomially convex and $P(K) = C(K)$. It is well known that no smoothness is needed in C^1 but that in higher dimensions further assumptions are required for the above conclusion. We have

THEOREM. *Let $f_1, f_2, \cdots, f_n \in C(I)$ separate the points of I and suppose that for $1 \leqq i \leqq n-1$, f_i is either $\mathcal{C}^1$ or real-valued. Then the separating uniform algebra which $f_1, f_2, \cdots, f_n$ generate is $C(I)$.*

If all the f_i, $1 \leqq i \leqq n-1$ are real valued, this theorem reduces to a result of Rudin [3]; on the other hand, if we consider the image K of I under $t \rightarrow (f_1(t), \cdots, f_n(t))$ we obtain a generalization of Stolzenberg's result on smooth arcs.

Applied to uniform algebras on the circle T, the methods of the previous theorems yield

THEOREM. *Let A be a separating uniform algebra on T which contains a function f which is locally* 1-1, *then either*

(i) *T is the maximal ideal space M_A of A, in which case $A=C(T)$ or*

(ii) *M_A-T is nonempty and has the structure of a 1-dimensional analytic space on which the functions in A are analytic.*

Finally we have the following which Shapiro and Shields [4] conjectured as an improvement of a result of Björk.

THEOREM. *Let $\Delta=\{z: |z|<1\}$. Let F be a closed subset of $\bar{\Delta}$ with $T\subseteq F\subseteq\bar{\Delta}$ such that*

(i) *F has no interior in $\mathbf{C}$,*

(ii) *$\Delta-F$ is connected.*

(iii) *$(\Delta\cap F)^-$ does not contain T.*

Let $g\in C(F)$ and suppose that the separating uniform algebra on F generated by g and z is a proper subalgebra of $C(F)$. Then there exists $G\in C(\bar{\Delta})$ such that

(i) *$G|T=g|T$,*

(ii) *G is analytic on $\Delta-F$.*

The proofs [1] will appear elsewhere, together with more complete references to the literature. J. E. Björk [2] has independently obtained similar results.

2. **A special case.** In order to indicate the methods, we prove the following special case of the first mentioned theorem.

PROPOSITION 1. *Let A be a separating uniform algebra on I which contains a function f which separates all but a finite number of pairs of points of I. Then $A=C(I)$.*

PROOF (SKETCH). It is easily seen that there are a finite number of functions in A which separate the points of I and so we may assume that A is finitely generated by $f_1=f, f_2, \cdots, f_n$. Let K be the homeomorphic image of I under the map $t\rightarrow(f_1(t), \cdots, f_n(t))$. Then K is an arc in $\mathbf{C}^n$ and z_1 (the first coordinate function) separates all but a finite number of pairs of points of K. Our goal is to prove $P(K)=C(K)$. We note that $\mathbf{C}-z_1(K)$ has finitely many components and in order to give a proof by induction on this number we prove a more general result.

PROPOSITION 2. *Let K be a finite disjoint union of arcs in $\mathbf{C}^n$. Suppose z_1 separates all but a finite number of pairs of points of K. Then $P(K)=C(K)$ and (hence) K is polynomially convex.*

PROOF. Let $L = z_1(K)$. $\mathbf{C}^1 - L$ has finitely many components. The proof will be by induction on this number k.

$k=1$: L does not separate the plane and L has no interior and so $P(L) = C(L)$. It follows that $z \to \bar{z}$ is in $P(L)$ and so $\bar{z} \circ z_1 = \bar{z}_1 \in P(K)$. It is easily seen from the Stone-Weierstrass theorem that $P(K)$ contains every $f \in C(K)$ which identifies the points that z_1 does. From this it follows that $P(K) = C(K)$.

Next we assume the result for $k-1$ and prove it for $k > 1$. Assume, for the moment, that K has been proved to be polynomically convex. Then $L = z_1(K)$ is the spectrum of z_1 as an element of $P(K)$. As $R(L) = C(L)$ it follows from the Gelfand theory that $F \circ z_1 \in P(K)$ for all $F \in C(L)$. In particular, $\bar{z}_1 \in P(K)$ and, as above, $P(K) = C(K)$.

It remains to show K is polynomially convex. Suppose not. Let Ω be a bounded component of $\mathbf{C}^1 - L$ such that there is an arc $\gamma \subseteq \partial\Omega$ which is also in the boundary of Ω_∞, the unbounded component of $\mathbf{C}^1 - L$. Let γ° denote γ with its endpoints deleted. We may assume z_1 is 1-1 on $z_1^{-1}(\gamma) \cap K$. Since γ is in the boundary of Ω_∞, $z_1^{-1}(\gamma) \cap K = z_1^{-1}(\gamma) \cap \hat{K}$ by [5]. Let $K_1 = K - z_1^{-1}(\gamma^\circ)$. Then K_1 satisfies the hypotheses of our proposition for the case $k-1$. So by the induction hypothesis, $P(K_1) = C(K_1)$ and K_1 is polynomially convex. We claim that $z_1(\hat{K}) \cap \Omega \neq \emptyset$. In fact if $p \in \hat{K} - K$, as K_1 is polynomially convex, there is a polynomial f such that $f(p) = 1 > \|f\|_{K_1}$. Let T be the component of $\{q \in \hat{K}: |f(q)| \geqq 1\}$ which contains p. Then by the local maximum modulus principle, T meets K; hence T meets $z_1^{-1}(\gamma^\circ) \cap K$. Hence $z_1(T)$ meets γ and so clearly $z_1(T)$ meets Ω.

Now by considering closed Jordan domains whose interiors are contained in Ω, whose boundaries contain γ and which meet $\Omega \cap z_1(\hat{K})$, it follows by [5] that $z_1^{-1}(\Omega) \cap \hat{K}$ is a 1-dimensional complex manifold in $z_1^{-1}(\Omega)$ which is mapped by z_1 biholomorphically onto Ω.

Let α be a straight line segment in Ω. Let γ_1 and γ_2 be arcs in $\Omega \cup \{\text{endpoints of } \gamma\}$ which join the endpoints of α to those of γ such that $\alpha \cup \gamma \cup \gamma_1 \cup \gamma_2$ is a Jordan curve bounding an open Jordan domain $W \subseteq \Omega$. Let $J = z_1^{-1}(\alpha) \cap \hat{K}$. J is a real analytic arc in $\mathbf{C}^n$. Let $X = (K - z_1^{-1}(\gamma)) \cup (z_1^{-1}(\gamma_1 \cup \gamma_2) \cap \hat{K})$. Then X is polynomially convex as it is a union of arcs such that $\mathbf{C}^1 - z_1(X)$ has $k-1$ components. By Stolzenberg's theorem $(X \cup J)\hat{} - X \cup J$ is a 1-dimensional analytic subset of $\mathbf{C}^n - X \cup J$. But by the local maximum modulus principle $(X \cup J)\hat{} = \hat{K} - z_1^{-1}(W \cup \gamma^\circ)$. It follows that $\hat{K} - K$ is a 1-dimensional analytic subset of $\mathbf{C}^n - K$. If $\lambda \in \Omega \cap z_1(\hat{K})$, then $z_1 - \lambda$ is an analytic function on $\hat{K}$ which has a zero on $\hat{K}$ and has a logarithm on K; this contradicts the argument principle [5]. We conclude that $\hat{K} = K$. Q.E.D.

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