EMBEDDING SPHERES AND BALLS IN CODIMENSION $\leqq 2$

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1. **Introduction.** In this note we announce some results on existence of PL embeddings of n-spheres and n-balls into a compact $(n-1)$-connected q-manifold ($n \geqq q-2$) by extending techniques of our preceding papers [5], [4]. Details will appear later. The result for locally flat embeddings with codimension two is satisfactory, although in general the low dimensional cases are still open.

By $\bigcup_{k=1}^r D_k^n$, $\bigcup_{k=1}^r S_k^n$ we denote the disjoint unions of r copies of the standard PL n-ball D^n, the standard PL n-sphere $S^n = \partial D^{n+1}$, resp. The embedding theorem of balls in codimension $\leqq 2$ is as follows:

THEOREM A. *Let Q be a compact $(n-1)$-connected PL q-manifold with nonempty boundary ∂Q.*

Let ϕ: $\bigcup_{k=1}^r D_k^n \to Q$ be a map such that $\phi(\bigcup_{k=1}^r S_k^{n-1}) \subset \partial Q$ and $\phi | \bigcup_{k=1}^r S_k^{n-1}$ is a PL embedding.

(I). *Suppose that one of the following holds.*

(0) $q = n \neq 3, 4$,

(1) $q = n+1 \neq 4$,

(2) $q = n+2 \neq 4$ *and* $r = 1$.

Then ϕ is homotopic to a proper PL embedding f: $\bigcup_{k=1}^r D_k^n \to Q$ keeping $\phi | \bigcup_{k=1}^r S_k^{n-1}$ fixed.

(II). *Suppose that $\phi | \bigcup_{k=1}^r S_k^{n-1}$ is locally flat, and that*

(1) $q = n+1 \neq 4$ *or*

(2) $q = n+2 =$ *odd and* $r = 1$.

Then ϕ is homotopic to a locally flat PL embedding f: $\bigcup_{k=1}^r D_k^n \to Q$ keeping $\phi | \bigcup_{k=1}^r S_k^{n-1}$ fixed.

(*Refer to* [13, *Chapter* 8, *Corollary* 5].)

In case $q-n=0$, Theorem A, (I) is equivalent to the generalized Poincaré conjecture. In case $q=n+1=4$, Theorem A is still open. In case $n=2$ and $Q=D^4$, refer to [13, Chapter 8, Counterexample 1].

In case $q=n+2=$ even, Theorem A, (II) is false because of the existence of nonslice knots ([1] and [6, Chapter III]).

The embedding theorem of spheres in codimension $\leqq 2$ is as follows:

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THEOREM B. *Let Q be a compact $(q-3)$-connected PL q-manifold. Suppose that $q\neq 4$.*

(I). *If $q\geqq 5$, a basis of $H_{q-1}(Q; Z)$ can be represented by mutually disjoint locally flat PL$(q-1)$-spheres. In particular, any element of $H_{q-1}(Q; Z_2)$ can be represented by a locally flat PL$(q-1)$-sphere.*

(II). *Any element of $H_{q-2}(Q; Z)$ can be represented by a PL$(q-2)$-sphere.*

(III). *Further, if $q=$odd, then any elements of $H_{q-2}(Q; Z)$ can be represented by a locally flat PL$(q-2)$-spheres. (Refer to* [**2**, *Corollary 1.2*].)

Theorem B, (I) is best possible by the homology reason. In case $q=4$, Theorem B, (I) and (II) are still open.

Theorem B, (III) is best possible because of the following.

THEOREM C (WITH RONNIE LEE). *For each even integer $n\geqq 2$, there exists a compact PL$(n+2)$-manifold Q which is an abstract regular neighborhood of S^n such that no nontrivial element of $H_n(Q; Z)$ can be represented by a locally flat PL n-sphere. (Refer to* [**7**].)

This is a modification of our preceding results [**5**, Theorem 2], whose proof may be improved to obtain the above by making use of Reidemeister torsions, which was pointed out to the author by Ronnie Lee.

As an implication of Theorems A and B we have the codimension $\leqq 2$ extension of Irwin's Theorem [**2**].

THEOREM D. *Let M and Q be compact PL m- and q-manifolds. Let ϕ: $(M, \partial M)\to(Q, \partial Q)$ be a map such that $\phi|\partial M$ is a PL embedding and $\phi^{-1}(\partial Q)=M$.*

Suppose that $m\geqq 5$, $q-m\leqq 2$ and

(1) M is $(2m-q)$-connected, and

(2) Q is $(2m-q+1)$-connected.

Then ϕ is homotopic to a proper PL embedding f: $M\to Q$ keeping ∂Q fixed.

We remark here that by the normal PL bundle theory for locally flat PL embeddings [**3**], [**11**], [**12**] and the so-called Cairns-Hirsch smoothing theory [**8**], the adjective "locally flat PL" in theorems can be replaced by "smoothable," if Q is smoothable.

2. **The structure of compact $(q-3)$-connected q-manifolds ($q\geqq 5$).** In the following, all things are considered from the piecewise linear viewpoint. Let Q be a compact $(q-3)$-connected q-manifold with nonempty boundary ∂Q. Suppose that $q\geqq 5$. Then by Poincaré-

Lefschetz duality and the universal coefficient theorem, $H_{q-1}(Q) \cong H^1(Q, \partial Q) \cong H_1(Q, \partial Q)$, $H_{q-2}(Q) \cong H^2(Q, \partial Q) \cong H_2(Q, \partial Q)$ are free of ranks α, β, where α, β are the Betti numbers of $H_{q-1}(Q)$, $H_{q-2}(Q)$, resp. Let $x=\{x_1, \cdots, x_\alpha\}$, $y=\{y_1, \cdots, y_\beta\}$ be given bases of $H_{q-1}(Q)$, $H_{q-2}(Q)$ and let $\bar{x}=\{\bar{x}_1, \cdots, \bar{x}_\alpha\}$, $\bar{y}=\{\bar{y}_1, \cdots, \bar{y}_\beta\}$ be corresponding bases of $H_1(Q, \partial Q)$, $H_2(Q, \partial Q)$ by the isomorphism above. From the general position we can represent these bases $\bar{x}$ and $\bar{y}$ by properly embedded arcs and disks having trivial normal bundles, since Q is 1-connected and $H_2(Q, \partial Q) \cong H_1(\partial Q)$. Let C be the complement of the union of open normal bundles of the disks and the arcs in Q. Then we have a handle decomposition; $Q=(\partial Q \times D) +(\bar{\phi}_1)+ \cdots +(\bar{\phi}_\alpha)+(\bar{\psi}_1)+ \cdots +(\bar{\psi}_\beta)+C$, where handles $(\bar{\phi}_l)$, $(\bar{\psi}_k)$ are just the trivial normal bundles of the arcs, disks representing $\bar{x}_l$, $\bar{y}_k$ and hence of indices 1, 2, resp. By looking at this decomposition upside down, we have the dual decomposition;

$$Q = C + (\psi_1) + \cdots + (\psi_\beta) + (\phi_1) + \cdots + (\phi_\beta),$$

where (ϕ_k), (ψ_l) are the duals to $(\bar{\phi}_k)$, $(\bar{\psi}_l)$ and of indices $(q-1)$, $(q-2)$, resp. Then the handles (ϕ_k), (ψ_l) represent j_*x_k, j_*y_l of $H_{q-1}(Q, C)$, $H_{q-2}(Q, C)$, where $j_*\colon \tilde{H}_*(Q) \to H_*(Q, C)$ is the natural homomorphism from the reduced homology group $\tilde{H}_*(Q)$ to $H_*(Q, C)$. Notice that handles (ϕ_k), (ψ_l) are mutually disjoint. Therefore, $H_*(Q, C)$ is torsion free and $j_*\colon \tilde{H}_*(Q) \to H_*(Q, C)$ is an isomorphism and $\tilde{H}_*(C)=0$. On the other hand by the general position C is 1-connected. Thus C is a compact contractible q-manifold.

Now we have proved the following

THEOREM 2.1. *Let Q be a compact $(q-3)$-connected q-manifold. Let α, β be the Betti numbers of $H_{q-1}(Q)$, $H_{q-2}(Q)$. Suppose that $q \geqq 5$. Given bases x, y of $H_{q-1}(Q)$, $H_{q-2}(Q)$, then we have a handle decomposition of Q relative to a compact contractible q-manifold C;*

$$Q = C + (\psi_1) + \cdots + (\psi_\beta) + (\phi_1) + \cdots + (\phi_\alpha)$$

such that handles (ϕ_k), (ψ_l) are mutually disjoint, of indices $q-1$, $q-2$ and represent the bases j_*x, j_*y, resp.

REMARK. For an $(n-1)$-submanifold M of ∂Q, if $q=n+1$ and $\beta=0$ or if $q=n+2$, then by the general position we may take the handle decomposition so that $M \subset \partial C$.

3. Embeddings of balls and spheres into a contractible manifold and a homology sphere. In codimension two case, the proof of Theorems A and B is based on the following special case of Theorem A which is an extension of results on knot cobordisms due to Kervaire [6] and Levine [9].

THEOREM 3.1. *Let C be a compact contractible q-manifold and let $\phi: \bigcup_{r=0}^{k} S_k^{n-1} \to \partial C$ be a locally flat embedding. Let Q be a manifold obtained from C by attaching r handles of index n via a framing of $\phi | \bigcup_{k=1}^{r} S_k^{n-1}$.*

Suppose that $q=n+2=2m+1$. Then $\phi | S_0^{n-1}: S^{n-1} \to 2Q$ extends to a locally flat embedding f: such that $f(D^n)$ meets the right-hand ball with algebraic intersection number 1.

Suppose that $q=n+2=2m+2 \geqq 6$. Then f can be taken to be locally flat.

In codimension one case, it is based on the following

LEMMA 3.2. *Suppose that $q \geqq 5$.*

(1) *Let C and C' be compact contractible q-manifolds with homeomorphic boundaries ∂C and $\partial C'$. Then a homeomorphism $h: \partial C \to \partial C'$ extends to a homeomorphism $H: C \to C'$.*

(2) *A homology $(q-1)$-sphere M bounds a contractible q-manifold.*

This may be well known and implies the following

THEOREM 3.4. *Let C be a compact contractible q-manifold and let $\phi: \bigcup_{k=1}^{r} S_k^{n-1} \to \partial C$ be an embedding. Suppose that*

$$q=n+1 \geqq 5.$$

Then ϕ extends to a proper embedding $f: \bigcup_{k=1}^{r} D_k^n \to C$.

Finally, the proof of Theorem A may be reduced to the locally flat case in virtue of the following

LEMMA 3.5. *Let M be a homology m-sphere, and let $f: S^n \to M$ be an embedding. Suppose that $(m, n) \neq (4, 2)$. Then f is isotopic to a locally flat embedding (perhaps by a locally knotted isotopy) keeping the complement of a given regular neighborhood of $f(S^n)$ in M fixed.*

This is a generalization of Fox-Milnor-Noguchi's Theorem [1], [10].

4. Applications: Some results on compact $(q-3)$-connected q-manifolds. An implication of Theorem 2.1 is the following generalization of [4, Theorem 3.11] and [5, Theorem 5].

THEOREM 4.1. *Let Q be a compact $(q-3)$-connected q-manifold with nonempty boundary and let α, β be the Betti numbers of $H_{q-1}(Q)$, $H_{q-2}(Q)$. Suppose that $q \geqq 5$.*

(1) *Then the boundary* ∂Q *of* Q *has the homology of a manifold obtained from some copies of* S^{q-1} *and* β *copies of* $S^{q-2}\times S^1$ *by taking* $\alpha+1$ *connected sums.*

(2) *Conversely, if a closed* $(q-1)$*-manifold* M *has the homology of the manifold above, then* M *bounds a compact* $(q-3)$*-connected* q*-manifold* Q *so that* α, β *are the Betti numbers of* $H_{q-1}(Q)$ *and* $H_{q-2}(Q)$.

(3) *Moreover, if* $(Q, \partial Q)$ *is oriented, then there are at most* 2β *distinct orientation preserving homeomorphism classes of oriented compact* $(q-3)$*-connected* q*-manifold* $(Q', \partial Q')$ *whose boundaries* $\partial Q'$ *are homeomorphic to* ∂Q *preserving orientations.*

(4) *In particular, if* $\beta=0$, *then an orientation preserving homeomorphism* $h: \partial Q\rightarrow Q'$ *extends to an orientation preserving homeomorphism* $H: Q\rightarrow Q'$.

Let Q be a compact q-manifold homotopy equivalent to S^n. We define an invariant $\omega(Q)\in Z_2$ as follows: $\omega(Q)=0$, if Q admits a locally flat embedding $f: S^n\rightarrow Q$ which is a homotopy equivalence, and $\omega(Q)=1$, otherwise. Note that $\omega(Q)=0$ if and only if a basis of $H_n(Q)$ can be represented by a locally flat n-sphere. In the situation above we have

THEOREM 4.2. (I). *The following statements are equivalent*:

(1) $\omega(Q)=0$.

(2) *Q can be embedded in* S^q.

(3) *Any embedding* $f: S^n\rightarrow \text{Int } Q$ *is isotopic to a locally flat embedding keeping the complement of a given regular neighborhood of* $f(S^n)$ *in* Q *fixed.*

(II). *In particular, if* $\omega(Q)=0$, *then* $Q\times D$ *is homeomorphic to* $S^n\times D^{q-n+1}$ *and the double of* Q *is homeomorphic to* $S^n\times S^{q-n}$.

The statements (I), (III) of Theorem B imply that $\omega(Q)=0$, provided $q=n+1\geqq 5$ or $q=n+2=\text{odd}$.

COROLLARY 4.3. *Let* Q *be a compact* q*-manifold homotopy equivalent to* S^n. *Suppose that* $q\geqq 5$ *and either* $q=n+1$ *or* $q=n+2=odd$. *Then all the statements of Theorem* 4.2 *hold.*

In case $q=4$, we have some weaker statements: Let Q be a compact 4-manifold. Suppose that Q collapses a 2-subpolyhedron L homeomorphic to the wedge $\vee_{k=1}^{\alpha} S_k^2$. We define an invariant $I(Q)$ in Z_2 as follows: $I(Q)=0$, if each 2-sphere of L has the self-intersection number a multiple of 2 and $I(Q)=1$, otherwise. Then we have

THEOREM 4.4. (I). *The following statements are equivalent*:

(1) $I(Q)=0$.

(2) *$Q\times D$ is homeomorphic to a boundary connected sum of α copies of $S^2\times D^3$.*

(3) *The double of Q is homeomorphic to a connected sum of α copies of $S^2\times S^2$.*

(II). *The following statements are equivalent:*

(1) $I(Q)=1$.

(2) *$Q\times D$ is homeomorphic to a boundary connected sum of $\alpha-1$ copies of $S^2\times D^3$ and the nontrivial D^3 bundle over S^2.*

(3) *The double of Q is homeomorphic to a connected sum of $(\alpha-1)$ copies of $S^2\times S^2$ and the nontrivial S^2 bundle over S^2.*

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