ON A NONTRIVIAL HIGHER EXTENSION OF REPRESENTABLE ABELIAN SHEAVES[1]

BY LAWRENCE S. BREEN

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Let $\mathcal{S}$ be the category of abelian sheaves in the *fppf* topology over a base scheme S, as defined in Demazure and Grothendieck [3, exposé IV §6.3]. This is an abelian category with enough injectives (see Artin [1, 1.6, 1.8]). For any F in $\mathcal{S}$ and any integer $i \geqq 0$, the functor $\mathrm{Ext}^i(F, -)$ from $\mathcal{S}$ to the category of abelian groups is defined in the usual manner to be the ith derived functor of the functor $\mathrm{Hom}(F, -)$. Let $S = \mathrm{Spec}(k)$ where k is a separably closed field of characteristic 2; we denote by α_2 the scheme $\mathrm{Spec}\,(k[x]/(x^2))$ with the usual group law (see for example Oort [8]), by G_m the multiplicative group scheme, and identify these objects of the category $\mathcal{C}$ of commutative algebraic group schemes over S with the objects in $\mathcal{S}$ which they represent. We show that $\mathrm{Ext}^2(\alpha_2, G_m) \neq 0$.

Via the identification just mentioned, $\mathcal{C}$ is a full subcategory of $\mathcal{S}$ which however does not contain enough injectives. It is nonetheless possible to define a functor Ext^i within the category $\mathcal{C}$. For $G, G' \in \mathcal{C}$, define $\mathrm{Ext}^i(G, G')$ to be the group of equivalence classes of i-fold Yoneda extensions in $\mathcal{C}$ of G by G'. This point of view, which was introduced by Serre in [9], was systematically developed by Oort in [8]. Oort shows in particular that $\mathrm{Ext}^i(H, G_m) = 0$ for $i \geqq 1$, where H is any finite group scheme over an algebraically closed groundfield. Our computation thus illustrates the fact that the two definitions of Ext^i are not equivalent.

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The technique used below in computing $\mathrm{Ext}^i(\alpha_2, G_m)$ (where henceforth we will always mean the first definition of Ext^i) is that of [2]; since only a small part of the theory described there is needed in our special case, we restate in detail the facts required.

Eilenberg and MacLane have defined [4, p. 659], [5] for every abelian group G a complex of free abelian groups $A(G)$ called the abelian complex of G:

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$$\begin{aligned}
A(G)_i &= 0 \qquad i \leqq 0\\
A(G)_1 &= \mathbf{Z}[G] \text{ is generated by elements } [x]\\
A(G)_2 &= \mathbf{Z}[G^2] \text{ is generated by elements } [x|y]\\
A(G)_3 &= \mathbf{Z}[G^3] \times \mathbf{Z}[G^2] \text{ is generated by elements } [x|y|z],\ [x|_2 y]\\
A(G)_4 &= \mathbf{Z}[G^4] \times \mathbf{Z}[G^3] \times \mathbf{Z}[G^3] \times \mathbf{Z}[G^2]\\
&\qquad \text{is generated by elements } [x|y|z|w],\ [x|y|_2 z],\ [x|_2 y|z],\ [x|_3 y]
\end{aligned}$$

where in each case x, y, z, w range over all elements of G. The boundaries ∂_i: $A(G)_i \to A(G)_{i-1}$ are defined on the generators and extended by linearity:

$$\begin{aligned}
&\partial_1 = 0\\
&\partial_2[x|y] = [x] - [x+y] + [y]\\
&\partial_3[x|y|z] = [y|z] - [x+y|z] + [x|y+z] - [x|y],\ \partial_3[x|_2 y] = [x|y] - [y|x]\\
&\partial_4[x|y|z|w] = [y|z|w] - [x+y|z|w] + [x|y+z|w] - [x|y|z+w] + [x|y|z]\\
&\partial_4[x|y|_2 z] = [x|_2 z] - [x+y|_2 z] + [y|_2 z] + [x|y|z] - [x|z|y] + [z|x|y]\\
&\partial_4[x|_2 y|z] = [x|_2 y] - [x|_2 y+z] + [x|_2 z] - [x|y|z] + [y|x|z] - [y|z|x]\\
&\partial_4[x|_3 y] = -[x|_2 y] - [y|_2 x].
\end{aligned}$$

The homology of this complex (which is just the stable homology of the Eilenberg-MacLane space) was computed in low dimensions by Eilenberg and MacLane [5] and in the general case by Cartan. In the lowest dimensions one has:

$$\begin{gathered}
\sigma\colon G \to H_1(A(G)) \qquad \sigma(x) = [x]\\
H_2(A(G)) = 0\\
\gamma\colon G/2G \to H_3(A(G));
\end{gathered}$$

γ is defined on a representative $x \in G$ of $G/2G$ by $\gamma(x) = [x|_2 x]$. One checks that $\gamma(2x)$ is a boundary, so γ passes to the quotient.

The complex $A(G)$ and the isomorphisms σ and γ are functorial in G, so one can define without difficulty, for any abelian presheaf P, a complex of abelian presheaves $A(P)$ with again $\sigma\colon P \to H_1(A(P))$, $H_2(A(P)) = 0$, $\gamma\colon P/2P \to H_3(A(P))$. Applying the functor "associated sheaf," which is exact [1, 1.6], one defines for any abelian sheaf F on some topology a complex of abelian sheaves $A(F)$ of the form $A(F)_1 = \mathbf{Z}[F]$, $A(F)_2 = \mathbf{Z}[F^2]$, etc. . . . , with $\sigma\colon F \to H_1(A(F))$, $H_2(A(F)) = 0$, $\gamma\colon F/2F \to H_3(A(F))$ where $F/2F$ is the cokernel of multiplication by 2 in the category of abelian sheaves (for more details, see [2]).

Examine now the special case where $F=\alpha_2$ (considered as an fppf sheaf over Spec (k)). In this case the complex $A(F)$ is G_m-acyclic: $\mathrm{Ext}^i(\boldsymbol{Z}[\alpha_2], G_m)=H^i_{fppf}(\alpha_2, G_m)$ since both are the value at G_m of the ith derived functor of the functor $\mathrm{Hom}(\boldsymbol{Z}[\alpha_2], -)=H^0_{fppf}(\alpha_2, -)$. Now, since G_m is smooth, $H^i_{fppf}(\alpha_2, G_m)=H^i_{et}(\alpha_2, G_m)$ (see [7, Appendix]) and this last group is trivial for i positive since the affine ring of α_2 is Henselian (in fact Artinian) with separably closed residue class field [1, 4.9].

In any abelian category with enough injectives, there is a well-known spectral sequence comparing the homology and the cohomology of an acyclic complex [6, p. 100]:

$$E_2^{p,q} = \mathrm{Ext}^p(H_q(A(\alpha_2), G_m) \Rightarrow H^{p+q}(A(\alpha_2); G_m)$$

where $H^j(A(\alpha_2); G_m)$ is the jth cohomology group of the complex $A^*(\alpha_2, G_m)$ with $A^q(\alpha_2, G_m)=\mathrm{Hom}(A(\alpha_2)_q, G_m)$ for all integers q, the coboundary maps being the obvious ones.

$E_2^{0,3}=\mathrm{Hom}(F/2F, G_m)=\mathrm{Hom}(\alpha_2, G_m)$ since multiplication by 2 is trivial on α_2. This last group is trivial as is well known (and can be checked easily). Since $H_q(A(\alpha_2))=0$ for $q\leqq 0$ and $q=2$, $E_2^{p,q}=0$ for those values of q. The spectral sequence thus degenerates in low dimensions. In particular

$$E_2^{2,1} = \mathrm{Ext}^2(\alpha_2, G_m) \approx H^3(A(\alpha_2); G_m).$$

We will exhibit a 3-cocycle which is not a coboundary. To simplify the notation we will, in all the following, write $k[x_1, \cdots, x_n]^*$ for the group of invertible elements in the ring

$$k[x_1, \cdots, x_n]/(x_1^2, \cdots, x_n^2)$$

(i.e. the group of truncated polynomials with nontrivial constant terms). A 3-cochain is an element of

$$\begin{aligned}\mathrm{Hom}(A(\alpha_2)_3, G_m) &= \mathrm{Hom}(\boldsymbol{Z}[\alpha_2^3] \times \boldsymbol{Z}[\alpha_2^2], G_m)\\ &= \mathrm{Hom}(\boldsymbol{Z}[\alpha_2^3], G_m) \oplus \mathrm{Hom}(\boldsymbol{Z}[\alpha_2^2], G_m)\\ &= H^0(\alpha_2^3, G_m) \oplus H^0(\alpha_2^2, G_m)\\ &= k[x, y, z]^* \oplus k[x, y]^*.\end{aligned}$$

A 3-cochain is thus a pair (f, g) with $f=f(x, y, z)\in k[x, y, z]^*$, $g=g(x, y)\in k[x, y]^*$. The cocycle condition for such a pair is $\partial(f, g)=(1, 1, 1, 1)$ where

$$\partial(f, g) = (f_1, f_2, f_3, f_4) \in k[x, y, z, w]^* \oplus k[x, y, z]^* \oplus k[x, y, z]^* \oplus k[x, y]^*$$

with

$$f_1(x, y, z, w)=f(y, z, w)f(x+y, z, w)^{-1}f(x, y+z, w)f(x, y, z+w)^{-1}f(x, y, z),$$
$$f_2(x, y, z)=g(x, z)g(x+y, z)^{-1}g(y, z)f(x, y, z)f(x, z, y)^{-1}f(z, x, y),$$
$$f_3(x, y, z)=g(x, y)g(x, y+z)^{-1}g(x, z)f(x, y, z)^{-1}f(y, x, z)f(y, z, x)^{-1},$$
$$f_4(x, y)=g(x, y)^{-1}g(y, x)^{-1}.$$

Similarly, (f, g) is a coboundary if there exists an element $h=h(x, y)\in k[x, y]^*$ such that

$$f(x, y, z) = h(y, z)h(x + y, z)^{-1}h(x, y + z)h(x, y)^{-1},$$
$$g(x, y) = h(x, y)h(y, x)^{-1}.$$

Consider the 3-cochain $(1, 1+uxy)\in k[x, y, z]^*\oplus k[x, y]^*$, where u is any nontrivial element of k. One checks without difficulty that this pair satisfies the cocycle condition above and is not a coboundary.

REMARKS. 1. It is immediate that if u and v are distinct elements of k, the pairs $(1, 1+uxy)$ and $(1, 1+vxy)$ are not cohomologous, so k (with the additive group law) is a subgroup of $\mathrm{Ext}^2(\alpha_2, G_m)$. In fact one can prove that $k\approx\mathrm{Ext}^2(\alpha_2, G_m)$.

2. More generally, if k is any separably closed field of characteristic $p\neq 0$, $\mathrm{Ext}^i(\alpha_p, G_m)=0$ for $0<i<2p-2$, $\mathrm{Ext}^{2p-2}(\alpha_p, G_m)\approx k$.

3. Let A be an abelian variety with $\mathrm{Hom}(\alpha_2, A)\neq 0$. One concludes that $\mathrm{Ext}^3(A, G_m)\neq 0$.

4. It is possible to give an explicit description of these nontrivial extensions as Yoneda extensions in $\mathcal{S}$ (involving of course some non-representable sheaves).

5. The cocyle exhibited above also gives a nontrivial element of $\mathrm{Ext}^2(\alpha_2, G_m)$ where we now identify the commutative group schemes with the objects in the category of abelian presheaves on Spec(k) which they represent and mean by Ext^2 the second derived functor of Hom in this category.

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Massachusetts Institute of Technology, Cambridge, Massachusetts 02139