ON ABSOLUTELY CONTINUOUS TRANSFORMATIONS

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1. **Introduction.** In this announcement, we examine absolutely continuous transformations T mapping the measure space (S, Σ, μ) onto the measure space (S', Σ', μ'). In order to obtain information about the change of measure induced by T, a weight function W' defined on $S' \times \mathfrak{D}$ is introduced, where $\mathfrak{D}$ is a certain subfamily of Σ. $W'(s', D)$ represents a weight assigned to the points in D which T maps into $s' \in S'$. We present structure theorems (Theorems 2 and 3) for weight functions which enable us to establish a transformation formula (Theorem 1) for integrals defined on the measure spaces. Theorem 1 includes all the existing transformation formulas for transformations which are absolutely continuous with respect to a real valued weight function. Moreover, the integrability condition necessary to ensure the existence of the formula is minimal, as we shall indicate in §3.

Rado and Reichelderfer [11] considered the case when the measure spaces are Euclidean n-space (both having the same dimension), with Lebesgue measurable sets and n-dimensional Lebesgue measure; T is a bounded continuous transformation defined on the bounded domain S. In particular, the weight function $\mu_e(s', T, D)$ generated by the topological index defined on indicator domains is used to define an essentially absolutely continuous transformation. Also the Banach indicatrix or crude multiplicity function $N(s', T, D)$ and the weight function $k(s', T, D)$ which counts the number of essential maximal model continua for (s', T, D) are treated in detail in [11]. In this classical setting, Craft [10] removed some conditions on the weight functions. Reichelderfer [13] developed a transformation theory for general measure spaces under certain standard hypotheses. Necessary and sufficient conditions were given in order that a transformation be absolutely continuous. In [14] it was shown that a large class of topological spaces satisfies these hypotheses; consequently, in this general topological setting the concepts of absolute continuity and generalized Jacobians can be effectively defined. Brooks [1], [3] developed the theory for integrals in Banach spaces and introduced a larger class of weight functions [2]; as a special case, signed weight functions may now be used when the spaces are oriented. Lebesgue decomposition theorems and measurability theorems for positive weight functions were considered by Chaney [6], [7], [8].

The theorems in §3 will be used at a later date to develop a chain rule for the product of absolutely continuous transformations. This problem was first solved for Euclidean n-space in [12] and was treated in a more general setting in [9].

2. **The setting.** In this section we establish notation and state some definitions. Throughout this paper we assume that the standard hypotheses for transformation theory H1–H8 are satisfied (see [13] or [1]). For the reader's convenience some of the families of sets occurring in these hypotheses are listed.

(S, Σ, μ) and (S', Σ', μ') are σ-finite complete measure spaces. T is a function (transformation) mapping S onto S'. $\mathfrak{D}$ is a subfamily of Σ containing $\varnothing$ and S. D will be a generic notation for a set in $\mathfrak{D}$. $T\mathfrak{D}\subseteq\Sigma'$ and the intersection of two sets in $\mathfrak{D}$ can be written as a countable union of disjoint sets from $\mathfrak{D}$. For every $E\in\Sigma$ and $\epsilon>0$ there exists a disjoint sequence $\{D_i\}$ such that $E\subseteq\bigcup D_i$ and $\mu(\bigcup D_i-E)<\epsilon$. A set $E\subseteq S$ is *$\mu\mu'$-null* if $E=A_1\cup A_2$, where $\mu(A_1)=\mu'(TA_2)=0$. $\Delta(D)$ ($\Delta^*(D)$) denotes the family of all finite (countable) collections of pairwise disjoint sets in $\mathfrak{D}$ contained in D. $\mathfrak{R}$ denotes the real numbers.

A *signed weight function* for T is a real valued function W' defined on $S'\times\mathfrak{D}$ such that:

(i) $W'(\cdot, D)=0$ a.e. μ' on $S'-TD$;

(ii) If $D_i\uparrow D$, then $\lim W'(\cdot, D_i)=W'(\cdot, D)$ a.e. μ';

(iii) If $\{D_i\}\in\Delta^*(D)$ and $D-\bigcup D_i$ is $\mu\mu'$-null, then $W'(\cdot, D)=\sum W'(\cdot, D_i)$ a.e. μ';

(iv) $W'(\cdot, D)$ is measurable for each D.

W' will always denote a signed weight function.

A *positive weight function* G' is a nonnegative function satisfying the above conditions except (iii) is replaced by the requirement that $\sum G'(\cdot, D_i)\leqq G'(\cdot, D)$ a.e. μ', whenever $\{D_i\}\in\Delta^*(D)$. Write $G_1'\prec G_2'$ if $G_1'(\cdot, D)\leqq G_2'(\cdot, D)$ a.e. μ' for every D.

T is of *bounded variation* with respect to $W'(BVW')$ if there exists a nonnegative function $K'\in\mathfrak{L}_1(\mu')$ such that for $\{D_i\}\in\Delta^*(S)$, $\sum|W'(\cdot, D_i)|\leqq K'$ a.e. μ'. T is *absolutely continuous* with respect to W' (ACW') if T is BVW' and there exists a function $f\in\mathfrak{L}_1(\mu)$ such that $\int_D f d\mu=\int_{TD} W'(\cdot, D)d\mu'$, for every D. f is called a *generalized Jacobian* for T relative to W' (it follows that f is unique in $\mathfrak{L}_1(\mu)$). A function g satisfies *condition* $(N)_T$ if $g=0$ a.e. μ on $T^{-1}E'$ whenever $\mu'(E')=0$. Let $W(D)=\int_{TD} W'(\cdot, D)\, d\mu'$, $D\in\mathfrak{D}$. Define

$$V(D, W) = \sup_{\Delta(D)} \sum |W(D_i)|\,;\; V'(\cdot, D) = \sup_{\Delta(D)} \sum |W'(\cdot, D_i)|\,.$$

The above definitions include the existing definitions of bounded variation and absolute continuity in the literature. We assume in the sequel that T is ACW' with generalized Jacobian f.

3. **The results.** Proofs for the theorems in this section will appear elsewhere. The main result is the following

THEOREM 1 (TRANSFORMATION FORMULA). *Let $H':S'\to\mathfrak{R}$ be measurable. Then $H'\circ Tf$ is measurable. If $H'\circ Tf$ is μ-integrable on a fixed set D, then $H'W'(\cdot, D)$ is μ'-integrable and*

$$\int_D H'\circ Tf d\mu = \int_{TD} H'W'(\cdot, D)d\mu'.$$

As mentioned above, this theorem extends the results of Rado and Reichelderfer in Euclidean n-space [11, p. 262]. We mention that the integrability of $H'W'(\cdot, D)$ does not imply the integrability of $H'\circ Tf$ on D [4]; however, the integrability of $H'V'(\cdot, D)$ implies the integrability of $H'\circ Tf$ on D [2]. The following results which are used to establish Theorem 1 are interesting in their own right.

THEOREM 2. *f satisfies condition $(N)_T$ and $V(D, W)=\int_D|f|d\mu$.*

The proof of the first assertion involves a technique similar to the one found in the proof of Lemma 3.1 in [2]. The proof of the second part is a long technical argument using γ-type partitions of elements of $\mathfrak{D}$ [13].

The next theorem yields a decomposition for weight functions which complements the pointwise decomposition theorems presented in [5].

THEOREM 3 (JORDAN DECOMPOSITION). *There exist positive weight functions Q', $Q'_\pm$ for T such that*

1. *$Q'=Q_+'+Q_-'$; $W'=Q_+'-Q_-'$.*
2. *T is ACQ', $ACQ'_\pm$; $|f|$ $(f^\pm)$ is generalized a Jacobian for T relative to Q' $(Q'_\pm)$.*
3. *If U', $U'_\pm$ are nonnegative real valued functions defined on $S'\times\mathfrak{D}$ such that $W'=U_+'-U_-'$ and $U'=U_+'+U_-'$ is a positive weight function for which T is ACU', then $U'_\pm$ are positive weight functions, T is ACU', and $U'_\pm\succ Q'_\pm$.*

In the proof, the function $\Psi_D(E')=\int_{(T^{-1}E')\cap D}|f|d\mu$, $E'\in\Sigma'$ is used (cf. [8]). Ψ_D is well defined on Σ' and $\Psi_D\ll\mu'$ by Theorem 2. Hence, we can define $Q'(\cdot, D)=(d/d\mu')\Psi_D$. $Q'_\pm$ are defined by 1. By using convergence theorems, one can show that Q', $Q_{\pm}'$ satisfy the conclusions of the above theorem.

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