ANOTHER THEOREM ON CONVEX COMBINATIONS OF UNIMODULAR FUNCTIONS

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Let R be a finite open Riemann surface; that is, R is obtained by deleting from a compact Riemann surface a finite number of disjoint closed discs, each of which has an analytic simple closed curve as boundary. Let $A(R)$ be the algebra of functions which are continuous on the closure of R and analytic on R; $A(R)$ is a Banach space under the supremum norm. An element f of $A(R)$ will be called *inner* if $|f|=1$ on the boundary of R. The following theorem extends the author's earlier result, where R was the unit disc in the complex plane [3].

THEOREM. *The closed convex hull of the inner functions in $A(R)$ is the unit ball of $A(R)$.*

The proof requires two lemmas whose proofs will be given after the proof of the theorem.

LEMMA 1. *Let $z_1, \cdots, z_N$ be distinct points of R and let h be an analytic function on R bounded by 1. Then there is an inner function f in $A(R)$ with $f(z_j)=h(z_j)$ for $j=1, \cdots, N$.*

LEMMA 2. *Let E be a compact subset of the boundary of R of zero harmonic measure and let μ be a positive regular Borel measure on E. If g is a continuous function on E of unit modulus, then there is a sequence $\{f_n\}$ of inner functions in $A(R)$ such that*

(i) *f_n converges to g a.e.μ and*

(ii) *f_n converges uniformly to one on compact subsets of R.*

PROOF OF THE THEOREM. Let Q be the closed convex hull of the inner functions in $A(R)$. By the basic separation theorem [2, V.2.10] if Q were not equal to the unit ball of $A(R)$, there would be a measure λ which strictly separated Q from some element of the unit ball of $A(R)$. By [1, Corollary 5] the set of linear functionals on $A(R)$ which attain their norm at some element of the unit ball of $A(R)$ is dense in the dual space of $A(R)$. Hence, it suffices to prove this: if λ is a measure on B, the boundary of R, with $\|\lambda\|=1=\int f d\lambda$, some $f\in A(R)$, $\|f\|=1$, then $\sup \{\operatorname{Re} \int q d\lambda : q \in Q\}=1$.

Such a measure λ has the form

$$d\lambda = \bar{f} g \, dm + \bar{f} \, d\mu$$

where m is harmonic measure for some fixed point p in R, μ is nonnegative and singular with respect to m, g is nonnegative, and the closed support of λ lies in the set where f has unit modulus. By choosing a sufficiently large compact subset of the support of μ we may also assume that the support of μ is compact and, of course, has zero m-measure.

If $z\in R$ let $P_z dm$ be the harmonic measure for z on B. It is easy to see that the linear span of the set $\{P_z : z\in R\}$ is dense in $L^1(B, m)$. Hence, given $\epsilon>0$, there are points $z_1, \cdots, z_N$ in R and constants $c_1, \cdots, c_N$ with $\| \sum_{i=1}^N c_i P_i - g\bar{f}\| < \epsilon$, where we have written P_i for P_{z_i}. Thus $0 \leqq \int g dm = \int f\bar{f} g dm \leqq \operatorname{Re}(\sum_{i=1}^N c_i f(z_i)) + \epsilon$.

By Lemma 1 there is an inner function I in $A(R)$ with $I(z_j) = f(z_j)$ for $j = 1, \cdots, N$. By Lemma 2, there is a sequence $\{f_n\}$ of inner functions in $A(R)$ with $f_n \to \bar{I} f$ a.e.μ and $f_n(z_j) \to 1$ for $j = 1, \cdots, N$. Let $h_n = I f_n$; then h_n is an inner function in $A(R)$ for each n, $h_n(z_j) \to f(z_j)$ for $j = 1, \cdots, N$ and $h_n \to f$ a.e.μ. Hence,

$$\begin{aligned}
\operatorname{Re}\int h_n d\lambda &= \operatorname{Re}\left(\int h_n \bar{f} g dm\right) + \operatorname{Re}\left(\int h_n \bar{f} d\mu\right) \\
&\geqq \operatorname{Re}\left(\int h_n \left(\sum_1^N c_i P_i\right) dm\right) - \epsilon + \operatorname{Re}\left(\int h_n \bar{f} d\mu\right) \\
&= \operatorname{Re}\left(\sum_1^N c_i h_n(z_i)\right) - \epsilon + \operatorname{Re}\left(\int h_n \bar{f} d\mu\right) \\
&\geqq \int g dm - 3\epsilon + \int d\mu - \epsilon = 1 - 4\epsilon
\end{aligned}$$

for n sufficiently large. This establishes the theorem.

PROOF OF LEMMA 1. This a result of Heins [4, p. 571].

PROOF OF LEMMA 2. By a theorem of Stout [7, Theorem IV. 1] there are three inner functions in $A(R)$, say h_1, h_2, and h_3, which separate the points of the closure of R and whose differentials have no common zero on R. $h_i(E)$ is a compact subset of the unit circle of arc length zero for $i = 1, 2, 3$. Let H embed R in the unit three-polydisc by $H(z) = (h_1(z), h_2(z), h_3(z))$, and let $F = H(E)$. Since F is a compact subset of $h_1(E) \times h_2(E) \times h_3(E)$, it is a peak-interpolation set for the polydisc algebra [6, Theorem 4.1]. Choose a function G in the polydisc algebra which is bounded by one and satisfies $G(H(z)) = g(z)$ for $z \in E$. Given $\epsilon > 0$ there are by Rudin's theorem [5, see final Remark] unimodular functions $U_1, \cdots, U_k$ in the polydisc algebra and positive numbers $\lambda_1, \cdots, \lambda_k$ which sum to 1 such that $\| \sum_1^k \lambda_j U_j - G \| < \epsilon$.

Let $V_j = U_j \circ H$. Then V_j is an inner function in $A(R)$ and $\|\sum_1^k \lambda_j V_j - g\|_E < \epsilon$. This implies that there are inner functions f_n in $A(R)$ such that $\int f_n \bar{g} d\mu \to 1$. (We are assuming that $\|\mu\| = 1$; this involves no loss of generality.) Hence

$$\int |f_n - g|^2 d\mu = \int |f_n|^2 + \int |g|^2 d\mu - 2 \operatorname{Re} \int f_n \bar{g} d\mu$$

$$= 2 - 2 \operatorname{Re} \int f_n \bar{g} d\mu \to 0.$$

Let $z_0 \in R$; by Lemma 1 there is an inner function I in $A(R)$ with $I(z_0) = 0$. By the above we can find inner functions f_n in $A(R)$ with f_n converging in $L^2(\mu)$ to $\bar{I} g$. Thus If_n is inner and in $A(R)$ for each n and If_n converges to g in $L^2(\mu)$ and vanishes at z_0. Finally, choose a sequence of numbers $\{\beta_n\}$ with $0 < \beta_n < 1$ and $\beta_n \to 1$. For each n there is an inner function g_n in $A(R)$ with

(i) $g_n(z_0) = 0$ and

(ii) the $L^2(\mu)$ distance from g_n to $(g - \beta_n)(1 - \beta_n g)^{-1}$ is less than $(1 - \beta_n)^2$.

If we put $f_n = (g_n + \beta_n)(1 + \beta_n g_n)^{-1}$, then f_n is in $A(R)$, is inner, has the value β_n at z_0, and finally, the $L^2(\mu)$ distance from g to f_n is no more than $2(1 - \beta_n)$. Since the f_n are bounded by 1 and converge to 1 at an interior point of R, they must converge uniformly to 1 on compact subsets of R. A subsequence converges a.e.μ to g.

References

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