# ON INTEGRAL REPRESENTATIONS

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Let $G$ be a finite group and $p$ a prime. $G$ is called cyclic mod $p$ if there exists a normal $p$-subgroup $N \trianglelefteq G$ such that $G/N$ is cyclic.

Let $R$ be a commutative ring with $1 \in R$. Write $\mathfrak{C}_R(G)$ for the set of subgroups $U \leqq G$, which are cyclic mod $p$ for some appropriate prime $p$ $(=p(U))$ with $pR \neq R$.

An $RG$-module $M$ is a finitely generated $R$-module, on which $G$ acts from the left by $R$-automorphisms. If $U \leqq G$ we write $M|_U$ for the $RU$-module, one gets by restricting the action of $G$ on the $R$-module $M$ to $U$.

If $N$ is an $RU$-module, we write $N^{U\to G}$ for the induced $RG$-module $RG \otimes_{RU} N$.

Two $RG$-modules $M$, $N$ are called weakly isomorphic, if there exists an $RG$-module $L$ and a natural number $k$, such that $k\cdot M \oplus L \cong k\cdot N \oplus L$ ($k\cdot M$ short for $M \oplus \cdots \oplus M$, $k$ times), we write then $M \simeq N$.

REMARK. If the Krull-Schmidt-Theorem holds for $RG$-modules, we have

$$M \simeq N \Leftrightarrow M \cong N.$$

THEOREM 1. *Let $M$, $N$ be two $RG$-modules. If $M|_U \simeq N|_U$ for all $U \in \mathfrak{C}_R(G)$, then $M \simeq N$. Moreover there exist for any $U \in \mathfrak{C}_R(G)$ two $R$-free $RG$-modules $M(U)$, $N(U)$ with $M(U)|_V \cong N(U)|_V$ for all $V \leqq G$, which do not contain any conjugate of $U$, but $M(U)|_U \not\simeq N(U)|_U$.*

One can get an even more precise statement by using Grothendieck-rings: Let $X(G, R)$ be the Grothendieck-ring of $RG$-modules with respect to split-exact sequences, i.e. $X(G, R)$ is an as additive group isomorphic to the free abelian group, generated by the isomorphism classes of $RG$-modules modulo the subgroup generated by all expressions of the form $M - M_1 - M_2$ with $M \cong M_1 \oplus M_2$—and the multiplication in $X(G, R)$ is given by the tensor-product $\otimes_R$ of $RG$-modules. Write $X_Q(G, R)$ for $Q \otimes X(G, R)$. Obviously $M \simeq N$ if and only if $M$ and $N$ represent the same element in $X_Q(G, R)$.

$X(\cdot, R)$ and $X_Q(\cdot, R)$ are obviously contravariant functors from the category of groups into the category of commutative rings. Especially for $U \leqq G$ one has restriction homomorphisms $\text{res}|_U$: $X(G, R) \to X(U, R)$, $X_Q(G, R) \to X_Q(U, R)$ and Theorem 1 reads now

THEOREM 1′. *$\prod_{U\in\mathfrak{C}_R(G)} \mathrm{res}|_U: X_Q(G, R)\to\prod_{U\in\mathfrak{C}_R(G)} X_Q(U, R)$ is injective.*

One can also describe the image of $X_Q(G,R)$ in $\prod_{U\in\mathfrak{C}_R(G)} X_Q(U,R)$. More generally let $\mathfrak{U}$ be any family of subgroups of $G$ closed with respect to subconjugation, i.e.

$$U, V \leqq G, \qquad g\in G, \qquad gVg^{-1}\leqq U\in\mathfrak{U}$$

implies $V\in\mathfrak{U}$. For any such triple $U$, $V\in\mathfrak{U}$ and $g\in G$ with $gVg^{-1}\leqq U$ one has a diagram:

$$\begin{array}{ccc} & \overset{\phi}{\nearrow} & X_Q(U,R) \\ X_Q(G,R) & & \downarrow \tau_g \\ & \underset{\psi}{\searrow} & X_Q(V,R) \end{array},$$

the maps $\phi$ and $\psi$ given by restriction, the map $\tau_g$ defined by $V\to U$, $v\to gvg^{-1}$, and one can easily see, that this diagram is commutative. Thus $\prod_{U\in\mathfrak{U}} \mathrm{res}|_U: X_Q(G, R)\to\prod_{U\in\mathfrak{U}} X_Q(U, R)$ maps $X_Q(G, R)$ into

$$X_Q(\mathfrak{U}, R) = \Big\{(x_U)_{U\in\mathfrak{U}}\in\prod_{U\in\mathfrak{U}} X_Q(U, R)\mid \tau_g(x_U) = x_V,$$

$$\text{whenever } U, V\in\mathfrak{U}, g\in G \text{ and } gVg^{-1}\leqq U\Big\}$$

and one has actually

THEOREM 2. *The canonical map $X_Q(G, R)\to X_Q(\mathfrak{U}, R)$ is always epimorphic and is an isomorphism if and only if $\mathfrak{U}\geqq\mathfrak{C}_R(G)$.*

It seems to be difficult, to prove a similar statement for $X(G,R)$ instead of $X_Q(G, R)$, but if $X'(G, R)$ denotes the image of $X(G, R)$ in $X_Q(G, R)$, i.e. $X(G, R)$ mod torsion, and if for any subconjugately closed family $\mathfrak{U}$ of subgroups of $G$ we write $\mathfrak{Y}\mathfrak{U}$ for $\{V\leqq G\mid$ there exists $U\trianglelefteq V$, $U\in\mathfrak{U}$, $V/U$ a $p$-group$\}$ then one can prove

THEOREM 3. *If $(x_V)_{V\in\mathfrak{Y}\mathfrak{U}}\in X'(\mathfrak{Y}\mathfrak{U}, R)\subseteq\prod_{V\in\mathfrak{Y}\mathfrak{U}} X'(V, R)$ then the projection $(x_U)_{U\in\mathfrak{U}}$ of $(x_V)_{V\in\mathfrak{Y}\mathfrak{U}}$ into $X'(\mathfrak{U}, R)\subseteq\prod_{U\in\mathfrak{U}} X'(U, R)$ is contained in the image of $X'(G, R)$ in $X'(\mathfrak{U}, R)$.*

REMARK. One can form a category $\mathfrak{U}$, whose objects are the subgroups in $\mathfrak{U}$ with $\mathrm{Hom}_{\mathfrak{U}}(V, U) = \{g\in G\mid gVg^{-1}\leqq U\}$ and obvious composition. Then $X(\cdot, R)$, $X_Q(\cdot, R)$, $X'$, $R)$ are contravariant functors on $\mathfrak{U}$ and one has

$$X(\mathfrak{U}, R) = \operatorname*{proj\,lim}_{\mathfrak{U}} X(\cdot, R), \qquad X_Q(\mathfrak{U}, R) = \operatorname*{proj\,lim}_{\mathfrak{U}} X_Q(\cdot, R),$$

$$X'(\mathfrak{U}, R) = \operatorname*{proj\,lim}_{\mathfrak{U}} X'(\cdot, R).$$

We will state one lemma, which is fundamental for the proof of the above theorems.

We say, that an $RG$-module $M$ is weakly, resp. quasi-$\mathfrak{U}$-induced, if there exists a natural number $k$ and for any $U \in \mathfrak{U}$ two $RU$-modules $N_1(U)$, $N_2(U)$ such that

$$k \cdot M \oplus \bigoplus_{U \in \mathfrak{U}} N_1(U)^{U \to G} \cong \bigoplus_{U \in \mathfrak{U}} N_2(U)^{U \to G},$$

$$\text{resp. } k \cdot \left(M \oplus \bigoplus_{U \in \mathfrak{U}} N_1(U)^{U \to G}\right) \cong k \cdot \left(\bigoplus_{U \in \mathfrak{U}} N_2(U)^{U \to G}\right).$$

For a $G$-set $S$ (i.e. a finite set, on which $G$ acts from the left by permutations) we write $R[S]$ for the free $R$-module, generated by $S$, considered as $RG$-module by extending the action of $G$ on the basis $S$ linearly to $R[S]$. Two $G$-sets $S_1$, $S_2$ are $\mathfrak{U}$-isomorphic ($S_1 \overset{\mathfrak{U}}{=} S_2$), if the restrictions $S_1|_U$ and $S_2|_U$ to any $U \in \mathfrak{U}$ are isomorphic. Then we have the following

LEMMA. *For a group $G$, a family $\mathfrak{U}$ of subgroups and a commutative ring $R$ the following four statements are equivalent:*

(i) *the trivial $RG$-module $R$ is weakly $\mathfrak{U}$-induced;*

(ii) *any $RG$-module is weakly $\mathfrak{U}$-induced;*

(iii) $X_Q(G, R) \to \prod_{U \in \mathfrak{U}} X_Q(U, R)$ *is injective;*

(iv) *if $S_1$, $S_2$ are two $\mathfrak{U}$-isomorphic $G$-sets, then* $R[S_1] \simeq R[S_2]$.

Any of these statements implies, that every $RG$-module is quasi-$\mathfrak{Y}\overline{\mathfrak{U}}$-induced with $\overline{\mathfrak{U}} = \{V \leqq G \mid \text{there exists } g \in G,\ U \in \mathfrak{U} \text{ with } gVg^{-1} \leqq U\}$, i.e. the subconjugate closure of $\mathfrak{U}$ and $\mathfrak{Y}\overline{\mathfrak{U}} = \{W \leqq G \mid \text{there exists } V \in \overline{\mathfrak{U}},\ V \trianglelefteq W,\ W/V \text{ a } p\text{-group}\}$ (just as before). Especially any $RG$-module is quasi-$\mathfrak{Y}\mathfrak{C}_R(G)$-induced—which generalizes a well-known result of Brauer-Bermann-Witt-Swan for the case $R = Q$. In case $\zeta$ is a $e$th root of unity with $e = \exp(G)$ and there is a homomorphism $Z[\zeta] \to R$ one can sharpen this result, to generalize Brauer's result on elementary subgroups. Define $\mathfrak{E}_R(G) = \{V \leqq G \mid \text{there exists } N \trianglelefteq V \text{ with } V/N \text{ elementary and } N \text{ a } p\text{-group for some } p \text{ with } pR \neq R\}$. Then any $RG$-module is quasi-$\mathfrak{E}_R(G)$-induced. One can also deduce intermediate statements, corresponding to the Bermann-Witt Theorem on $K$-elementary subgroups. There is still another way to generalize the above theorems: For any family $\mathfrak{U}$ of subgroups of $G$

define an $RG$-module $M$ to be $\mathfrak{U}$-projective, if $M$ is a direct summand in $\bigoplus_{U\in\mathfrak{U}} (M|_U)^{U\to G}$. One can develop a theory of $\mathfrak{U}$-projective $RG$-modules completely analogous to D. G. Higman's theory in the case $\mathfrak{U}=\{U\}$ and one can also define for any $RG$-module $M$ its family of vertices—corresponding to Green's theory, i.e. for any $RG$-module $M$ there exists a family of subgroups $\mathfrak{U}(M)$ such that $M$ is $\mathfrak{W}$-projective if and only if $\overline{\mathfrak{W}}\geqq\mathfrak{U}(M)$ ($\overline{\mathfrak{W}}$ as before the subconjugate closure of $\mathfrak{W}$) and all subgroups in $\mathfrak{U}(M)$ are $p$-groups for various primes $p$ with $pR\neq R$. And one can prove that two $\mathfrak{U}$-projective $RG$-modules $M$ and $N$ are weakly isomorphic, if their restrictions $M|V$ and $N|V$ are weakly isomorphic for all $V\leqq G$ which contain a normal $p$-subgroup $N\trianglelefteq V$ with $N\in\overline{\mathfrak{U}}$, $V/N$ cyclic and $pR\neq R$. In fact one proves Theorem 1 by using the above statement in some kind of complete induction argument, starting with $\mathfrak{U}=\{E\}$, the trivial subgroup. There are corresponding generalizations of the other statements.

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