AN ALGEBRAIC DUALIZATION OF FUNDAMENTAL GROUPS

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This note presents a construction of a Hopf algebra $\pi^1(A)$ for a given augmented commutative algebra A equipped with a derivation. Such a Hopf algebra may be taken as a dualized algebraic analogy of a fundamental group.

1. The construction of $\pi^1(A)$ is motivated by dualizing the fundamental group $\pi_1(X)$ of a differentiable manifold X with a base point x_0. Let A be the R-algebra of C^∞ functions on X equipped with the derivation d, which is the usual differentiation from A into the A-module $M=\Omega A$ of C^∞ 1-forms on X. Recall that the shuffle algebra $\mathrm{Sh}(M)$ consists of the R-module of the tensor algebra $T_R(M)$ and the shuffle multiplication $\circ$. We make $\mathrm{Sh}(M)$ a Hopf R-algebra with the comultiplication ζ: $\mathrm{Sh}(M)\to\mathrm{Sh}(M)\otimes\mathrm{Sh}(M)$ given by

$$w_1 \otimes \cdots \otimes w_r \mapsto \sum_{0\leq i\leq r} (w_1 \otimes \cdots \otimes w_i) \otimes (w_{i+1} \otimes \cdots \otimes w_r)$$

$\forall w_1, \cdots, w_r \in M$. Moreover the Hopf algebra $\mathrm{Sh}(M)$ possesses an antipode (or conjugation) j.

Denote by G the monoid of piecewise smooth loops of X with the base point x_0 under the equivalence relation of reparametrization. The monoid algebra RG is a Hopf algebra whose comultiplication Δ is given by $\Delta\alpha=\alpha\otimes\alpha$, $\forall\alpha\in G$.

Given a loop α: $[0, 1]\to X$, let $\int_\alpha w_1$ be the usual integral, and define, for $r>1$, iterated path integrals

$$\int_\alpha w_1 \cdots w_r = \int_0^1 \left(\int_{\alpha|[0,t]} w_1 \cdots w_{r-1}\right) w_r(\alpha(t), \dot{\alpha}(t))dt.$$

Then there is a pairing $\mathrm{Sh}(M)\times RG\to R$ such that

$$\langle w_1 \otimes \cdots \otimes w_r, \alpha\rangle = \int_\alpha w_1 \cdots w_r.$$

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This is actually a pairing of Hopf algebras so that, $\forall u, v \in \mathrm{Sh}(M)$, $\forall \alpha, \beta \in G$,

$$\langle u \circ v, \alpha \rangle = \langle u, \alpha \rangle \langle v, \alpha \rangle = \langle u \otimes v, \Delta\alpha \rangle,$$
$$\langle u, \alpha\beta \rangle = \langle \zeta u, \alpha \otimes \beta \rangle.$$

Moreover

$$\langle u, \alpha^{-1} \rangle = \langle ju, \alpha \rangle.$$

In the above pairing, $\mathrm{Sh}(M)$ is too large and has an ideal I which is orthogonal to RG. The ideal I is spanned by elements of the type

$$u \otimes fw \otimes v - (u \circ df) \otimes w \otimes v - f(x_0) u \otimes w \otimes v,$$

$\forall u, v \in \mathrm{Sh}(M)$, $f \in A$, $w \in M$. (See [1].) Let $P(A)$ denote the augmented quotient R-algebra $\mathrm{Sh}(M)/I$. Set $\bar{u} = u + I$.

Observe that the 1-form $df \in M = T^1(M) \subset T(M)$ can be taken as an element of $\mathrm{Sh}(M)$ so that $\langle df, \alpha \rangle = \int_\alpha df = 0$. Let $Q(A)$ denote the augmented quotient R-algebra of $P(A)$ over the ideal generated by all $\overline{df}, f \in A$. It turns out that $Q(A)$ is a quotient Hopf R-algebra of $\mathrm{Sh}(M)$. The induced pairing $Q(A) \times RG \to R$ is again that of Hopf algebras.

Let α_s, $0 \leqq s \leqq 1$, represent a piecewise smooth homotopy of loops from the loop α_0 to the loop α_1 at the base point x_0. Our interest is in those elements u of $Q(A)$ such that $\langle u, \alpha_s \rangle$ is independent of s. For this purpose, we devise algebraically a derivation δ of the R-algebra $Q(A)$ such that, if $\delta u = 0$, then

$$(d/ds)\langle u, \alpha_s \rangle = (d/ds) \int_{\alpha_s} u = 0.$$

Denote by $\pi^1(A)$ the kernel of the derivation δ, which happens to be a Hopf R-algebra with an antipode. Obviously there is an induced pairing of Hopf algebras

$$\pi^1(A) \times R\pi_1(X) \to R.$$

In our construction described below, the algebra $P(A)$ will be, in addition equipped with a derivation.

2. All rings and algebras will be commutative and possessing one. Let K be the ground ring. We begin with the category C of K-modules and shall only make use of the fact that C is a cocomplete tensored category, i.e. a tensored category (see [4]) which has tensor-product-preserving direct limits.

Denote by C' the morphism category of C. An object M of C' is a morphism of K-modules $d_M\colon |M| \to \Omega M$; and a morphism of C', $\phi\colon M \to M'$, consists of two morphisms of K-modules $|\phi|\colon |M| \to |M'|$ and $\Omega\phi\colon \Omega M \to \Omega M'$ with $d_{M'}|\phi| = \Omega\phi d_M$. The tensor product $M \otimes_{C'} M'$ is defined to be

$$(d_M \otimes 1, 1 \otimes d_{M'})\colon |M| \otimes |M'| \to \Omega M \otimes |M'| \oplus |M| \otimes \Omega M'.$$

Let $K^{\langle 1\rangle}$ denote the object $K \to 0$ in C'. With the ground object $K^{\langle 1\rangle}$ and the tensor product $\otimes_{C'}$, we have a cocomplete tensored category C'. A C'-algebra A can be identified with a K-algebra $|A|$ equipped with a derivation $d_A\colon |A| \to \Omega A$. If A is augmented, the augmentation of A will be denoted by ϵ_A.

3. A C'-algebra B is said to be exact if the sequence of K-modules

$$0 \to K \to |B| \xrightarrow{d_B} \Omega B \to 0$$

is exact. As in [1], we construct, for a given augmented C'-algebra A, an augmented exact C'-algebra $P(A)$ and a morphism of augmented C'-algebras $\chi_A\colon A \to P(A)$ which is universal with respect to morphisms from A to augmented exact C'-algebras. Moreover, there is a quotient augmented K-algebra $Q(A)$ of $|P(A)|$ such that the projection $\rho\colon |P(A)| \to Q(A)$ is universal with respect to the property that $\rho|\chi_A|$ is the composition

$$|A| \xrightarrow{|\epsilon_A|} K \to Q(A).$$

It happens that $Q(A)$ is a Hopf K-algebra with commutative multiplication. (In [1], $P(A)$ and $Q(A)$ are respectively denoted by $\mathrm{Sh}(d, p)$ and $\mathrm{Shc}(d, p)$, where $d = d_A$ and $p = |\epsilon_A|$.)

4. A C'-algebra A is said to be full if the composition

$$|A| \otimes \mathrm{Im}\, d_A \to |A| \otimes \Omega A \xrightarrow{m} \Omega A$$

is epic, where m denotes the scalar multiplication of the $|A|$-module ΩA.

Construct the cocomplete tensored category $C'' = (C')'$. Each object M in C'' is then a square diagram in C which can be transposed. The transposed object will be denoted by M^t.

Given an augmented full C'-algebra A, there is an augmented C''-algebra $\{A\}$ which is universal with respect to the property that $|\{A\}| = |\{A\}^t| = A$.

Observe that the functors P and Q for the category C' can be similarly constructed for the category C''. Furthermore, $|Q(\{A\})| = Q(A)$. Thus $Q(\{A\})$ represents a derivation δ of $Q(A)$.

We define $\pi^1(A)$ to be the kernel of the derivation δ. If $\otimes$ is also left exact, then $\pi^1(A)$ is a Hopf subalgebra of $Q(A)$ and possesses an antipode.

THEOREM. *If A is an augmented full C'-algebra and if d_A induces an exterior derivation $d_1: \Omega A \to \Omega A \wedge_{|A|} \Omega A$, then the composite morphism of K-modules*

$$\Omega A \xrightarrow{\Omega\chi_A} > \Omega P(A) \xrightarrow{i} |P(A)| \xrightarrow{\rho} Q(A)$$

induces a morphism of K-modules

$$\theta: H^1(A) = \operatorname{Ker} d_1/\operatorname{Im} d_A \to \pi^1(A),$$

where i is the morphism of K-modules which splits the exact sequence of the exact C'-algebra $P(A)$ in the same way as the augmentation of $P(A)$ does.

REMARK 1. If A is the augmented R-algebra equipped with the derivation as described in §1 and if the manifold X is compact, then it follows from de Rham's theorem that θ is injective.

REMARK 2. If X is a compact Riemannian surface, we let K be the complex number field and denote by A the augmented K-algebra of complex valued C^∞ functions on X equipped with the usual differentiation. Let $M' \subset M = \Omega A$ be the K-module of abelian differentials of the first kind. It can be shown that the image of the composition

$$\operatorname{Sh}(M') \to \operatorname{Sh}(M) \xrightarrow{\rho'} Q(A)$$

lies in $\pi^1(A)$, where ρ' denotes the projection. This can be roughly interpreted to mean that the Hopf algebra $\pi^1(A)$ contains a Hopf subalgebra consisting of iterated path integrals (from x_0) of abelian differentials of the first kind.

5. Two morphisms of augmented C'-algebras $\phi_i: A \to A'$, $i = 0, 1$, are said to be homotopic if there exists an exact C'-algebra B with two augmentations s_0, s_1 and a morphism of C'-algebras $\Phi: A \to A' \otimes_{C'} B$ such that $\phi_i = (1 \otimes s_i)\Phi$, $i = 0, 1$, and $(\epsilon_{A'} \otimes 1)\Phi$ is the composition

$$A \xrightarrow{\epsilon_A} K^{(1)} \to B.$$

(This is an algebraic analogy of a base point preserving homotopy. See [2].)

THEOREM. *If ϕ_0 and ϕ_1 are homotopic, the* $\pi^1(\phi_0) = \pi^1(\phi_1)$.

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