MANIFOLDS OF THE HOMOTOPY TYPE OF (NON-LIE) GROUPS

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Hilbert's Fifth Problem implies that a topological group which is topologically a finite dimensional manifold is a Lie group. Until quite recently, the only topological groups of the homotopy type of compact manifolds known were Lie groups. In 1963 Slifker exhibited a topological group of the homotopy type of S^3 yet not multiplicatively equivalent to SU(1). In 1968, Hilton and Roitberg announced the discovery of a 10-dimensional manifold M_7^{10} which admits a multiplication yet is not of the homotopy type of a Lie group. In fact, they showed $M_7^{10}\times S^3=\mathrm{Sp}\ (2)\times S^3$. They left open the question: Does M_7^{10} admit a homotopy associative multiplication, a necessary condition for M^{10} to be of the homotopy type of a topological group? We answer the question affirmatively; thus a homotopy version of Hilbert's Fifth Problem is false.

THEOREM 1. *There is a topological group G of the homotopy type of a compact manifold M^{10} (the 3-sphere bundle over S^7 described by Hilton and Roitberg) which is not of the homotopy type of any Lie group.*

More precisely we show the following

THEOREM 2. *Let $S^3\to M_n^{10}\to S^7$ be the principal S^3-bundle classified by $n\omega\in\pi_6(S^3)$, $n\in Z_{12}$, ω chosen as a generator such that the corresponding M_1^{10} is* Sp (2).

M_n^{10} is of the homotopy type of a Lie group if and only if $n\equiv\pm 1$ (12).

M_n^{10} is of the homotopy type of a topological group if $n\equiv\pm 1, \pm 5$ (12).

M_n^{10} admits a multiplication if $n\not\equiv 2$ (4).

The first part results from the classification of such bundles up to homotopy type and the classification of Lie groups. The case $n\equiv -1$ is realized by $\overline{\mathrm{Sp}}$ (2), the opposite symplectic group, which has the same underlying space as Sp (2) but the opposite order of multiplication.

The remainder of the theorem is proved using a new technique of Zabrodsky's called "mixing homotopy types" [2].

Let P be the set of primes and $P=P_1\cup P_2$, a decomposition into disjoint subsets. Let $\mathcal{C}P_1$ denote the class of abelian groups of orders not divisible by primes in P_2 and let $\mathcal{C}P_2$ denote the class of abelian groups not divisible by primes in P_1.

Let X, X_0 be simply connected CW-complexes.

THEOREM 3. *Let $f\colon X \to X_0$ be a rational homotopy equivalence. There exists a space $X(P_1)$ and a factorization $X \xrightarrow{f_1} X(P_1) \xrightarrow{f_2} X_0$ of f such that the fibre of f_i has homotopy groups belonging to $\mathcal{C}P_i$.*

If X, X_0 are H-spaces and f an H-map, then $X(P_1)$ is an H-space and f_2, f_1 are H-maps.

THEOREM 4. *Let X_i be simply connected* CW*-complexes for $i=0, 1, 2$. Let $f_i\colon X_i \to X_0$ be a rational homology equivalence.*

There exists a space X and maps $g_i\colon X \to X_i(P_i)$ such that the fibre has homotopy groups belonging to $\mathcal{C}P_{i\pm 1}$. If the "ingredients" X_i, f_i are H-spaces and H-maps, then X is an H-space and the maps g_i are H-maps.

If the ingredients are topological groups and homomorphisms, X has the homotopy type of a topological group.

Theorem 3 can be proved by constructing a modified Moore-Postnikov system for f in which the primary components of the homotopy groups $\pi_i(X_0, X)$ are put in first for $p \in P_1$ and then for $p \in P_2$. More simply, since f is a rational equivalence, its fibre F has the homotopy type of a product $\prod_{p\in P} F_p$ where F_p has p-primary homotopy only. $X(P_1)$ can be thought of as the subfibration of X in which the fibre is cut down to $\prod_{p\in P_1} F_p$.

To obtain the H-conditions, the following specific details are helpful. $X(P_1)$ can be constructed by a succession of principal $K(Z_p, n)$-fibrations, $p \in P_1$ induced by cohomology classes in the kernel of the cohomology morphism mod p.

LEMMA. *Let $f\colon X \to X^1$ be an H-map. If $f^*\colon H^i(X^1, Z_p) \to H^i(X, Z_p)$ is an isomorphism for $i < n-1$, monomorphism for $i = n-1$, and $\alpha \in \mathrm{Ker}\, f^* \cap H^n(X^1, Z_p)$ then α is represented by an H-map. The fibration Y induced over X^1 by α is therefore an H-space such that f can be lifted to an H-map $X \to Y$.*

That α is represented by an H-map follows from the fact that $f^*\alpha = 0$ is represented by an H-map and the obstructions lie where $(f \times f)^*$ is an isomorphism. The vanishing of these obstructions can thus be achieved in terms of chains whose images in X^2 are specified so the lifting of f is immediate.

Elements in the cokernel of the cohomology morphism are added by trivial principal (Z_{p^r}, n)-fibrations.

Theorem 4 is proved by taking X to be the fibre product (pull back) of $f_i\colon X_i(P_i) \to X_0$:

$$\begin{array}{ccc} X & \rightarrow & X_1(P_1) \\ \downarrow & & \downarrow \\ X_2(P_2) & \rightarrow & X_0 \end{array}$$

If the ingredients are topological groups and homomorphisms, Browder observed the construction can also be carried out in terms of BX_i to produce a classifying space Y. We then have X of the homotopy type of ΩY and hence of the homotopy type of a topological group by Milnor's constructions, if all spaces are countable CW.

PROPOSITION. *If X_1, X_2 are simply connected finite complexes, then X has the homotopy type of a finite complex.*

PROOF. Since $H^*(X_i; Q)$ is finite dimensional as a Q-vector space so is $H^*(X; Q)$. Since $H^*(X_i; Z_p)$ is finite dimensional for each p, $i=1, 2$, so is $H^*(X; Z_p)$. Moreover, the finite dimension has a common finite upper bound for Q and all p simultaneously (i.e. the maximum for X_1, X_2). Thus X has the homotopy type of a finite complex, for example that obtained by a homology decomposition of X.

We are now ready for examples. Let $\overline{\text{Sp}}$ (2) denote the "opposite symplectic group," i.e. the symplectic group obtained by multiplying quaternions in the opposite order. If $\omega \in \pi(S^3)$ is chosen as the generator which classifies $S^3 \rightarrow \text{Sp}(2) \rightarrow S^7$, then $\overline{\text{Sp}}$ (2) is classified by $-\omega$. Recall that $\pi_6(S^3) \approx Z_4 + Z_3$ with generators ν', α [1]. We have $\omega = \nu' + \alpha$. If we mix the homotopy type of $X_1 = \text{Sp}$ (2) and $X_2 = \overline{\text{Sp}}$ (2) over $K(Z, 3) \times K(Z, 7)$ with $2 \in P_2$ and $3 \in P_1$, the resulting group is the Hilton-Roitberg example, for $-\nu' + \alpha = 7\omega$. Interchanging Sp (2) and $\overline{\text{Sp}}$ (2) gives a group classified as a bundle by 5ω.

If we take $X_1 = \text{Sp}$ (2) and $X_2 = S^3 \times S^7$ with $2 \in P_1$ and $3 \in P_2$ then the bundle classified by $\nu' = 9\omega$ is seen to admit a multiplication but not a homotopy associative one [$\mathcal{P}^1$: $H^3 \rightarrow H^7$ is trivial which contradicts the nontriviality of cup cubes in the projective 3-spaces for X]. The same holds for 3ω.

If we interchange the roles of Sp (2) and $S^3 \times S^7$ then we see the bundles classified by $\pm 4\omega = \pm\alpha$ admit multiplications. The multiplication may be homotopy associative, although $S^7(\{2\})$ admits no homotopy associative multiplication.

REFERENCES

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