NONCLASSICAL SIMPLE LIE ALGEBRAS[1]

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Introduction. Let Φ be an algebraically closed field of characteristic $p>3$. In addition to the finite dimensional classical simple Lie algebras [12] over Φ a number of families of finite dimensional nonclassical simple Lie algebras over Φ have been discovered [1]–[3], [5]–[9], [13]. Until recently no general connection has been known between these algebras and any family of Lie algebras over fields of characteristic 0.

Recently Kostrikin and Shafarevitch [11] have given a unified construction of all known finite dimensional nonclassical simple restricted Lie algebras over Φ. These algebras are obtained as the analogues in prime characteristic of the simple infinite Lie algebras of Cartan type over $\boldsymbol{C}$.

We give here a generalization of the Kostrikin-Shafarevitch construction which gives all known finite dimensional nonclassical simple (not necessarily restricted) Lie algebras over Φ, as well as some which are new.[2]

I. **Definition of Lie algebras of Cartan type.** The infinite Lie algebras of Cartan type are certain Lie algebras over **C** which arise in the study of pseudogroups [10], [15]. They are characterized by the following conditions:

(1) L has a decreasing filtration $L=L_{-1}\supset L_0\supset L_1\supset \cdots$.

(2) $\bigcap L_i=(0)$.

(3) $[L_i, L_j]\subseteq L_{i+j}$ for $-1\leqq i, j$ (where $L_{-2}=L$).

(4) If $x\in L_i$ and $x\notin L_{i+1}$ for some $i\geqq 0$ then there exists $y\in L$ such that $[xy]\notin L_i$.

(5) $\dim L_{-1}/L_0<\infty$.

(6) $\dim L=\infty$.

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[1] These results are contained in the author's doctoral dissertation written under the guidance of Professor G. B. Seligman at Yale University. The author was a National Science Foundation Graduate Fellow at Yale.

[2] *Added in proof.* In a recent paper (*Graded Lie algebras of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1969), 251–322) Kostrikin and Shafarevitch have also studied the nonrestricted case and have obtained results which substantially overlap those of this paper.

The simple infinite Lie algebras of Cartan type over C have been determined [4], [10], [15]. They are:

$$\mathcal{W}(m) = \operatorname{Der} \mathbf{C}[[x_1, \cdots, x_m]], \qquad m \geqq 1.$$

$$\mathcal{S}(m) = \{D \in \mathcal{W}(m) \mid D\omega = 0, \omega = dx_1 \wedge \cdots \wedge dx_m\}, \qquad m \geqq 2.$$

$$\mathcal{V}(2r) = \left\{D \in \mathcal{W}(2r) \mid D\omega = 0, \omega = \sum_{i=1}^{r} dx_i \wedge dx_{i+r}\right\}, \qquad r \geqq 2.$$

$$\mathcal{R}(2r+1) = \left\{D \in \mathcal{W}(2r+1) \mid D\omega = u\omega, u \in \mathbf{C}[[x_1, \cdots, x_{2r+1}]],\right.$$

$$\left.\omega = dx_{2r+1} + \sum_{i=1}^{r} x_i dx_{i+r} - x_{i+r} dx_i\right\}, \qquad r \geqq 1.$$

(The action of a derivation D on the algebra $\mathfrak{D}$ of differential forms is that of the Lie derivative [16, p. 92]. Thus $\mathfrak{D}$ is the exterior algebra on $\{dx_1, \cdots, dx_m\}$ over $\mathbf{C}[[x_1, \cdots, x_m]]$, $df = \sum (\partial f/\partial x_i) dx_i$, $D(df) = d(Df)$, $D(f\omega) = (Df)\omega + f(D\omega)$, and $D(\omega \wedge \eta) = D\omega \wedge \eta + \omega \wedge D\eta$ for all $f \in \mathbf{C}[[x_1, \cdots, x_m]]$ and all ω and $\eta \in \mathfrak{D}$.)

We now consider certain $\boldsymbol{Z}$-subalgebras of these algebras. Define $A(m) = \{\alpha: \{1, \cdots, m\} \to \boldsymbol{Z} \mid \alpha(i) \geqq 0 \text{ for } 1 \leqq i \leqq m\}$. Define $\epsilon_i \in A(m)$ by $\epsilon_i(j) = \delta_{ij}$. For $\alpha, \beta \in A(m)$ define

$$\alpha! = \prod \alpha(i)!, \qquad \binom{\alpha}{\beta} = \prod \binom{\alpha(i)}{\beta(i)}$$

and

$$|\alpha| = \sum \alpha(i).$$

Let $\mathcal{A}(m) = \mathbf{C}[[x_1, \cdots, x_m]]$. For $\alpha \in A(m)$ define $x^\alpha = \prod x^{\alpha(i)}/\alpha(i)! \in \mathcal{A}(m)$. Set $\overline{\mathcal{A}}(m) = \{\sum a_\alpha x^\alpha \mid a_\alpha \in \boldsymbol{Z}\} \subset \mathcal{A}(m)$ where the summation extends over all $\alpha \in A(m)$ and infinite sums are allowed. Then $\overline{\mathcal{A}}(m)$ is a $\boldsymbol{Z}$-subalgebra of $\mathcal{A}(m)$. Set $\overline{\mathcal{W}}(m) = \operatorname{Der} \overline{\mathcal{A}}(m) = \{\sum f_i(\partial/\partial x_i) \mid f_i \in \overline{\mathcal{A}}(m), 1 \leqq i \leqq m\}$. Then $\overline{\mathcal{W}}(m)$ is a $\boldsymbol{Z}$-subalgebra of $\mathcal{W}(m)$. Now let Φ be an arbitrary field and define $\mathfrak{A}(m) = \overline{\mathcal{A}}(m) \otimes_Z \Phi$ and $W(m) = \overline{\mathcal{W}}(m) \otimes_Z \Phi$. Then $\mathfrak{A}(m)$ is an associative algebra over Φ with multiplication defined by bilinearity and

$$x^\alpha x^\beta = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta}$$

and $W(m) = \{\sum f_i D_i \mid f_i \in \mathfrak{A}(m)\}$ (where D_i is the image of $\partial/\partial x_i$) is a Lie algebra of derivations of $\mathfrak{A}(m)$. The action of $W(m)$ on $\mathfrak{A}(m)$ is given by

$$D_i x^\alpha = x^{\alpha-\epsilon_i}$$

and multiplication in $W(m)$ is given by

$$[x^\alpha D_i, x^\beta D_j] = \binom{\alpha+\beta-\epsilon_i}{\alpha} x^{\alpha+\beta-\epsilon_i} D_j - \binom{\alpha+\beta-\epsilon_j}{\beta} x^{\alpha+\beta-\epsilon_j} D_i.$$

Define $\bar{\mathcal{S}}(m) = \mathcal{S}(m) \cap \overline{\mathcal{W}}(m)$ and $S(m) = \bar{\mathcal{S}}(m) \otimes_Z \Phi$. Define $V(2r)$ and $R(2r+1)$ similarly. Now $W(m)$, $S(m)$, $V(2r)$, and $R(2r+1)$ are infinite dimensional Lie algebras over Φ. We now consider certain finite dimensional subalgebras of them.

Define $A(n_1, \cdots, n_m) = \{\alpha \in A(m) \mid \alpha(i) < p^{n_i},\ 1 \leqq i \leqq m\}$ (where characteristic $\Phi = p$). Then $\mathfrak{A}(n_1, \cdots, n_m) = \langle x^\alpha \mid \alpha \in A(n_1, \cdots, n_m)\rangle$ is a subalgebra of $\mathfrak{A}(m)$. Hence $W(n_1, \cdots, n_m) =$ the stabilizer in $W(m)$ of $\mathfrak{A}(n_1, \cdots, n_m)$ is a subalgebra of $W(m)$.

Let $\mathfrak{A}(m)_i = \{ \sum a_\alpha x^\alpha \mid a_\alpha = 0$ unless $|\alpha| \geqq i+1\}$. Then $\mathfrak{A}(m)$ is a topological algebra with topology defined by taking $\{\mathfrak{A}(m)_i \mid i \geqq 0\}$ to be a base of neighborhoods of 0. For $1 \leqq i \leqq r$ define $\mathfrak{A}_i(2r) = \{ \sum a_\alpha x^\alpha \mid a_\alpha = 0$ if $\alpha(j) \neq 0$ for some $j \neq i$ or $i+r\}$.

Let ϕ be an automorphism of $\mathfrak{A}(m)$. If $D \in W(m)$ then $D^\phi = \phi D \phi^{-1}$ is a derivation of $\mathfrak{A}(m)$. Following Ree [13] we say that ϕ is an admissible automorphism of $\mathfrak{A}(m)$ (with respect to $W(m)$) if ϕ is continuous and $W(m)^\phi \subseteq W(m)$.

LEMMA 1. *If ϕ is an admissible automorphism of $\mathfrak{A}(m)$ then* $\det(D_i \phi x^{\epsilon_j})$ *is a unit in* $\mathfrak{A}(m)$.

DEFINITION 1. A Lie algebra L over a field Φ of characteristic $p > 0$ is a Lie algebra of Cartan type if $L = K''$ where K is one of the following algebras:

(7) $W(n_1, \cdots, n_m)$ where $\sum (p^{n_i} - 1) > 2$.

(8) $S(m)^\phi \cap W(n_1, \cdots, n_m)$ where $m \geqq 2$, $\sum (p^{n_i} - 1) > 3$, ϕ is an admissible automorphism of $\mathfrak{A}(m)$ and $a^{-1} D_i a \in \mathfrak{A}(n_1, \cdots, n_m)$ for $1 \leqq i \leqq m$ where $a = \det(D_i \phi x^{\epsilon_j})$.

(9) $V(2r)^\phi \cap W(n_1, \cdots, n_{2r})$ where $r \geqq 2$, ϕ is an admissible automorphism of $\mathfrak{A}(2r)$, $\det(D_i \phi x^{\epsilon_j}) \in \mathfrak{A}(n_1, \cdots, n_{2r})$ and ϕ stabilizes $\mathfrak{A}_i(2r)$ for $1 \leqq i \leqq r$.

(10) $R(2r+1) \cap W(n_1, \cdots, n_{2r+1})$ where $r \geqq 1$, $p > 2$.

II. **Simplicity of Lie algebras of Cartan type.** Since the infinite Lie algebras of Cartan type possess filtrations satisfying conditions (1)–(6) it is not surprising that Lie algebras of Cartan type possess filtrations with similar properties. We use these properties to prove the simplicity of Lie algebras of Cartan type.

DEFINITION 2. A Lie algebra L is said to be strongly filtered with respect to M if L satisfies (1)–(5), if $L_2 \neq (0)$ and if M is a subspace of L_1 such that $L_i \subseteq [L, L_{i+1}] + L_{i+1} + M$ for all $i \geqq 0$.

LEMMA 2. *If L is a finite dimensional Lie algebra which is strongly filtered with respect to M and $M^{(n)} = (0)$ then $L^{(n)}$ is simple.*

THEOREM 1. *Any Lie algebra of Cartan type is simple.*

In view of Lemma 2 to prove Theorem 1 it is sufficient to show that each of the algebras K in (7)–(10) is strongly filtered with respect to a subspace M such that $M^{(2)} = (0)$. This is done separately for each of the four cases.

III. Identification of known algebras.

THEOREM 2. *If Φ is an algebraically closed field of characteristic $p > 3$ then every known finite dimensional nonclassical simple Lie algebra over Φ is of Cartan type.*

We prove this by a case by case analysis of the known algebras (those listed in [**14**, pp. 105–110]). In the course of the proof we obtain the following complete classification of the generalized Witt algebras [**13**].

COROLLARY. *If Φ is an algebraically closed field of characteristic $p > 0$ then any generalized Witt algebra over Φ is isomorphic to some $W(n_1, \cdots, n_m)$. If $W(n_1, \cdots, n_m) \cong W(r_1, \cdots, r_s)$ then $m = s$ and $n_i = r_{\sigma(i)}$ for $1 \leqq i \leqq m$ where σ is a permutation of $1, \cdots, m$.*

IV. New simple Lie algebras. Computation shows that if $n + 3 \not\equiv 0 \pmod p$ and $n = \sum n_i$ then $R(2r+1) \cap W(n_1, \cdots, n_{2r+1})$ is a simple Lie algebra of dimension p^n. If $p > 3$ its derivation algebra has dimension $p^n + n - (2r+1)$. By comparing these dimensions with those for the known simple Lie algebras we prove

THEOREM 3. *If $p > 3$, $n + 3 \not\equiv 0 \pmod p$, $n > m \geqq 3$ where m is an odd integer and $m \neq p^s + s$ for any integer s then $R(m) \cap W(n_1, \cdots, n_m)$ is a new simple Lie algebra.*

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