THREE-MANIFOLDS WITH FUNDAMENTAL GROUP A FREE PRODUCT

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1. **Introduction.** The purpose of this paper is to announce some results concerning the structure of a compact 3-manifold M (possibly with boundary) where $\pi_1(M)$ is a free product. Related questions for M closed have been considered in [**1**], [**2**], [**4**], [**6**], [**8**].

We use the term *map* to mean continuous function. If M is a manifold, we use Bd M and Int M to stand for the boundary and interior of M, respectively. The disk D is said to be *properly embedded* in the 3-manifold M if

$$D \cap \text{Bd } M = \text{Bd } D.$$

The compact 3-manifold H_n is called a *handlebody of genus n* if H_n is the regular neighborhood of a finite connected graph having Euler characteristic $1-n$.

The combinatorial terminology is consistent with that of [**9**]. The terms in group theory may be found in [**3**]. Furthermore, all maps and spaces are assumed to be in the PL category.

2. **Bounded Kneser Conjecture.**

THEOREM 2.1. *Let M denote a compact 3-manifold with nonvoid boundary where $\pi_1(M) \approx A * B$, a free product. Then there is a compact 3-manifold M' with nonvoid boundary so that*

(i) *M' has the same homotopy type as M, and*

(ii) *there is a disk D' properly embedded in M' where $M'-D'$ consists of two components M_1 and M_2 with $\pi_1(M_1) \approx A$ and $\pi_1(M_2) \approx B$.*

OUTLINE OF PROOF. Let K_A and K_B denote CW-complexes with $\pi_1(K_G) \approx G$ and $\pi_n(K_G)=0$, $n \geqq 2$, $G=A, B$. Let p denote a point not in $K_A \cup K_B$. Define $\overline{K}_A$ and $\overline{K}_B$ as the mapping cylinders of maps from p into K_A and K_B, respectively. Let K denote the CW-complex obtained by identifying the copy of p in $\overline{K}_A$ with the copy of p in $\overline{K}_B$. It follows that $\pi_1(K) \approx A * B$ and $\pi_n(K)=0$, $n \geqq 2$ (see [**1**, p. 669]).

There is a simplicial map f of M into K (K may be chosen so that any finite collection of cells in K has a simplicial subdivision) so that f_* is an isomorphism of $\pi_1(M)$ onto $\pi_1(K)$.

LEMMA A. *Let M, K, f, p be as above. There is a map g: $M \rightarrow K$ so that*

(i) *g is homotopic to f relative to a base point of $\pi_1(M)$, and*

(ii) *each component of* $g^{-1}(p)$ *is a disk properly embedded in* M.

If $g^{-1}(p)$ has only one component, then we are finished. Otherwise, we continue in the following way.

For M, K, and p as above, the map $g: M \to K$ is said to be *reduced* if each component of $g^{-1}(p)$ is a disk properly embedded in M. If g is reduced, the *complexity of* g, $\#(g)$, is the number of components of $g^{-1}(p)$. Using this notation we wish to find a compact 3-manifold M' having the same homotopy type as M and a reduced map $g': M' \to K$ with $\#(g')=1$ and g'_* an isomorphism.

We are able to find a handlebody, H_n, of genus n in M and 3-cells $B_1, \cdots, B_k$ so that for each $i=1, \cdots, k$,

$$B_i \cap H_n \subset \text{Bd } B_i \cap \text{Bd } H_n = A_i$$

is an annulus, $B_i \cap g^{-1}(p)=\emptyset$, $B_i \cap B_j=\emptyset$, $i \neq j$, and $M=H_n \cup B_1 \cup \cdots \cup B_k$. Under these conditions we say $(H_n, B_1, \cdots, B_k)$ is a *Heegaard splitting for* M *relative to* g.

A path γ in Bd H_n is called a *binding tie* (see [7]) if $\gamma \cap g^{-1}(p)$ consists of the endpoints of γ, the endpoints of γ are in distinct components of $g^{-1}(p)$, and the loop $g(\gamma)$ based at p is contractible in K (actually, since $g(\gamma) \subset \overline{K}_1$ or $\overline{K}_2$, $g(\gamma)$ is contractible in $\overline{K}_1$ or $\overline{K}_2$).

LEMMA B. *If* M, g, p *are as in Lemma* A *and* $\#(g)>1$, *then there is a Heegaard splitting* $(H_n, B_1, \cdots, B_k)$ *of* M *relative to* g *and a nonsingular (i.e. an arc) binding tie in* Bd H_n.

We are now able to obtain a compact 3-manifold M' having the same homotopy type as M and a map

$$g': M' \to K$$

so that

(i) g'_* is an isomorphism of $\pi_1(M')$ onto $\pi_1(K)$,

(ii) g' is reduced,

(iii) $\#(g')=\#(g)$, and

(iv) there is a Heegaard splitting $(H'_n, B'_1, \cdots, B'_k)$ of M' relative to g' and a nonsingular binding tie γ' in Bd H'_n so that $\gamma' \cap B'_i=\emptyset$ for each $i=1, \cdots, k$.

LEMMA C. *If* M', b', γ', p *are as above, then there is a map* $h': M' \to K$ *so that*

(i) h' *is homotopic to* g' *relative to a base point for* $\pi_1(M')$,

(ii) h' *is reduced, and*

(iii) $\#(h')<\#(g')$.

The following example shows that we cannot replace "homotopy type" by "homeomorphic" in part (i) of the previous theorem.

Let

$$N = S^1 \times S^1 \times I$$

where S^1 is the 1-sphere and I the unit interval. Let

$$T_0 = S^1 \times S^1 \times \{0\} \quad \text{and} \quad T_1 = S^1 \times S^1 \times \{1\}.$$

Choose disks $D_0 \subset T_0$ and $D_1 \subset T_1$. Let B denote the unit 3-cell. Choose disks D_0' and D_1' in Bd B so that $D_0' \cap D_1' = \emptyset$. Let h denote an orientation reversing homeomorphism from the pair (D_0', D_1') to (D_0, D_1). Define

$$M = N \underset{h}{\cup} B,$$

the 3-manifold obtained from the disjoint union of N and B by identifying x with $h(x)$, $x \in D_0' \cup D_1'$.

It is an elementary exercise using Van Kampen's Theorem to see that

$$\pi_1(M) \approx (Z \times Z) * Z$$

where Z is the infinite cyclic group. The 3-manifold M is irreducible; i.e. M is *irreducible* if each polyhedral two-sphere in M bounds a 3-cell in M. Using this fact we obtain

PROPOSITION 2.2. *If M is the 3-manifold constructed above and D is a disk properly embedded in M so that $M - D$ consists of two components M_1 and M_2, then the closure of either M_1 or M_2 is a 3-cell.*

COROLLARY 2.3. *If M is a compact 3-manifold and*

$$\pi_1(M) \approx A * B,$$

then there are compact 3-manifolds M_1 and M_2 so that $\pi_1(M_1) \approx A$ and $\pi_1(M_2) \approx B$.

3. Splitting along a disk.

THEOREM 3.1. *Let M denote a compact 3-manifold with nonvoid, connected boundary. Suppose that each disk D properly embedded in M separates M. If*

$$\pi_1(M) \approx A * B,$$

then there is a disk D properly embedded in M so that $M - D$ consists of two components M_1 and M_2 with $\pi_1(M_1) \approx A$ and $\pi_1(M_2) \approx B$.

OUTLINE OF PROOF. We proceed much along the same lines as in Theorem 2.1 through Lemma B of that theorem. Hence, borrowing from the notation of Theorem 2.1, suppose $g: M \to K$ is a map so that g_* is an isomorphism and g is reduced. We assume $g^{-1}(p)$ has more than one component (if $g^{-1}(p)$ has only one component, then we are finished).

Let $(H_n, B_1, \cdots, B_k)$ be a Heegaard splitting for M relative to g. Suppose γ is a nonsingular binding-tie in Bd H_n.

LEMMA D. *Using the above notation, there is a map* $h: M \to K$ *so that*
(i) *h is homotopic to g relative to a base point for* $\pi_1(M)$,
(ii) *h is reduced, and*
(iii) $^{\#}(h) < {}^{\#}(g)$.

THEOREM 3.2. *Let M denote a compact 3-manifold with nonvoid, connected boundary. If* $\pi_1(M) \approx Z * G$, *then there is a disk D properly embedded in M so that $M-D$ is connected and* $\pi_1(M-D) \approx G$.

PROOF. It is sufficient to show that there is a disk D properly embedded in M so that $M-D$ is connected (see the remarks on p. 27, vol. II of [3]).

If each disk properly embedded in M separates M, then by Theorem 3.1 there is a disk D properly embedded in M so that $M-D$ has two components M_1 and M_2 with $\pi_1(M_1) \approx Z$ and $\pi_1(M_2) \approx G$. However, we have

LEMMA E. *If M_1 is a compact 3-manifold with nonvoid boundary and* $\pi_1(M_1) \approx Z$, *then there is a disk D properly embedded in M_1 so that $M_1 - D$ is connected.*

We conclude in any case that the desired disk D may be obtained in M.

A group G is said to be *freely reduced* if whenever $G \approx G_1 * G_2$ then neither G_1 nor G_2 is a free group.

THEOREM 3.3. *Let M denote a compact 3-manifold with nonvoid, connected boundary. Suppose* $\pi_1(M) \approx A * B$, *a free product, and* $\pi_1(M)$ *is freely reduced. Then there is a disk D properly embedded in M so that $M-D$ consists of two components M_1 and M_2 with* $\pi_1(M_1) \approx A$ *and* $\pi_1(M_2) \approx B$.

4. Another proof of Kneser's Conjecture. For an account of Kneser's Conjecture see [5] especially §§3, 12, 15, 17, and 20. Also, see [1], [2], [6]. Whitehead [8] was the first to obtain a satisfactory proof of Kneser's Conjecture.

The group G is said to be *indecomposable* if whenever $G \approx G_1 * G_2$, then either G_1 or G_2 is the trivial group. If

$$G \approx A_1 * \cdots * A_n$$

where each A_i is indecomposable we call $A_1 * \cdots * A_n$ a *decomposition* of G. Each finitely generated group has a decomposition which is unique up to order and isomorphism [3].

KNESER CONJECTURE. *Let M denote a closed 3-manifold. Then*

$$\pi_1(M) \approx A * B,$$

a free product, iff there is a polyhedral 2-sphere S in M so that $M-S$ consists of two components M_1 and M_2 with $\pi_1(M_1) \approx A$ and $\pi_1(M_2) \approx B$.

We obtain Kneser's Conjecture as a corollary to

THEOREM 4.1. *Suppose M is a closed 3-manifold and*

$$\pi_1(M) \approx A_0 * A_1 * \cdots * A_k$$

is a decomposition of $\pi_1(M)$. Then there is a mutually exclusive collection of polyhedral 2-spheres $S_1, \cdots, S_k$ in M so that $M - \bigcup_1^k S_i$ consists of $k+1$ components $N_0, \cdots, N_k$ with $\pi_1(N_i) \approx A_i$ for $0 \leqq i \leqq k$.

PROOF. We proceed by induction on the integer k where $k \geqq 1$. The general situation is no more difficult than the situation $k=1$.

Let C denote a 3-cell in M. Let $M' = M - \text{Int } C$. Then $\pi_1(M') \approx \pi_1(M)$. There are two cases to consider.

Case 1. $\pi_1(M)$ is not freely reduced.

We choose the notation in this case so that $A_0 \approx Z$, the infinite cyclic group. Applying Theorem 3.2 we find a disk D properly embedded in M' so that $M' - D$ is connected. Hence, we are able to find a 2-sphere S in M so that $D \subset S$ and $M - S$ is connected. We have

LEMMA F. *Let M denote a closed 3-manifold and suppose S is a 2-sphere in M so that $M-S$ is connected. Then there is a 2-sphere S_1 in M so that $M - S_1$ consists of two components M_0 and $\overline{M}$ where $\pi_1(M_0) \approx Z$.*

The inductive hypothesis applies to the 3-manifold obtained from $\overline{M}$ by sewing a 3-cell B_1 onto S_1 along Bd B_1.

Case 2. $\pi_1(M)$ is freely reduced.

We proceed as in Case 1 only now we apply Theorem 3.3.

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