ON RELATIVE GROTHENDIECK RINGS

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Communicated by Hyman Bass, April 21, 1969

Let K be a field of characteristic $p(\neq 0$, to exclude trivial cases) and let G be a finite goup. A KG-module M is a finite dimensional K-vector space, on which G acts K-linearly from the left.

The Green ring $a(G)$ of G(w.r.t. K) is the free abelian group, spanned by the isomorphism classes of indecomposable KG-modules, with the multiplication induced from the tensor product $\oplus_K$ of KG-modules (see [4]).

If $U \leqq G$ one has a restriction map $a(G) \to a(U)$, induced from restricting the action of G on a KG-module M to U, thus getting a KU-module $M|_U$. Let $\mathfrak{U}$ be a family of subgroups of G. An exact sequence

$$E: 0 \to M' \to M \to M'' \to 0$$

is said to be $\mathfrak{U}$-split, if

$$E|_U: 0 \to M'|_U \to M|_U \to M''|_U \to 0$$

is a split exact sequence of KU-modules for any $U \in \mathfrak{U}$.

For any $\mathfrak{U}$-split exact sequence E of KG-modules define $\chi_E = M - M' - M''$ to be its Euler characteristic in $a(G)$. Write $i(G, \mathfrak{U})$ for the linear span of the elements $\chi_E \in a(G)$, where E runs through all $\mathfrak{U}$-split exact sequences of KG-modules. $i(G, \mathfrak{U})$ is an ideal in $a(G)$ and $a(G, \mathfrak{U}) = a(G)/i(G, \mathfrak{U})$ the Grothendieck ring of G relative to $\mathfrak{U}$ (see [1], [6]).

LEMMA 1. *Let $\mathfrak{U}_1$, $\mathfrak{U}_2$ be two families of subgroups of G. Then the multiplication map $a(G) \times a(G) \to a(G)$ sends $i(G, \mathfrak{U}_1) \times i(G, \mathfrak{U}_2)$ into $i(G, \mathfrak{U}_1 \cup \mathfrak{U}_2)$.*

PROOF. If E_i: $0 \to M_i' \to M_i \to M_i'' \to 0$ is exact and $\mathfrak{U}_i$-split, then the tensor product of these two complexes E_1, E_2 is exact and $\mathfrak{U}_1 \cup \mathfrak{U}_2$-split, therefore $\chi_{E_1 \otimes E_2} = \chi_{E_1} \cdot \chi_{E_2} \in i(G, \mathfrak{U}_1 \cup \mathfrak{U}_2)$.

An KG-module M is $\mathfrak{U}$-projective, if M is a direct summand in $\oplus_{U \in \mathfrak{U}} (M|_U)^{U \to G}$ (see [3]), where for a KU-module N we write $N^{U \to G}$ for the induced KG-module $KG \otimes_{KU} N$.

Write $k(G, \mathfrak{U})$ for the linear span of the $\mathfrak{U}$-projective modules in $a(G)$. The canonical epimorphism $a(G) \twoheadrightarrow a(G, \mathfrak{U})$ induces a map κ: $k(G, \mathfrak{U}) \to a(G, \mathfrak{U})$, which has also been called the Cartan map (see [1], [6], [7]).

THEOREM 1. *The Cartan map* κ: $k(G, \mathfrak{U}) \to a(G, \mathfrak{U})$ *is monic and its cokernel is a p-power-torsion-module.*

COROLLARY 1. *All torsion in* $a(G, \mathfrak{U})$ *(if there is any) is p-power-torsion.*

PROOF. If $\mathrm{Tor}(a(G, \mathfrak{U})) = T$ is the torsion submodule of $a(G, \mathfrak{U})$, then $T \cap \kappa(k(G, \mathfrak{U})) = 0$ because κ is injective and $k(G, \mathfrak{U}) \subseteq a(G)$ torsion free. Thus T maps injectively into cokernel κ. Now define $\mathfrak{Z}\mathfrak{U} = \{ V \leqq G \mid V_p \trianglelefteq V,\ V/V_p$ cyclic and $V_p \subseteq U$ for some $U \in \mathfrak{U}\}$, V_p the p-Sylow subgroup of V. For $V \in \mathfrak{Z}\mathfrak{U}$ the restriction map $a(G) \to a(V)$ factors over $a(G, \mathfrak{U})$.

THEOREM 2. *The kernel of the product of the restriction maps:*

$$a(G, \mathfrak{U}) \to \prod_{V \in \mathfrak{Z}\mathfrak{U}} a(V)$$

is exactly $\mathrm{Tor}(a(G, \mathfrak{U}))$.

COROLLARY 2. *If G is a p-group, then the product of the restriction maps* $a(G, \mathfrak{U}) \to \sum_{U \in \mathfrak{U}} a(U)$ *is exactly* $\mathrm{Tor}(a(G, \mathfrak{U}))$.

These results generalize some part of the more complete results, obtained in a more special case, in [**8**, Theorems 4, 5, 6].

COROLLARY 3. (a) *If M is* $\mathfrak{U}$*-projective, then there exists a number n and KV-modules* $M_1(V)$, $M_2(V)$ $(V \in \mathfrak{Z}\mathfrak{U})$ *with*

$$\underbrace{M \oplus M \oplus \cdots \oplus M}_{(n\text{-times})} \oplus \bigoplus_{V \in \mathfrak{Z}\mathfrak{U}} M_1(V)^{V \to G} \cong \bigoplus_{V \in \mathfrak{Z}\mathfrak{U}} M_2(V)^{V \to G}.$$

(b) *If* M_1, M_2 *are two* $\mathfrak{U}$*-projective KG-modules, then* $M_1 \cong M_2$ *if and only if* $M_1|_V \cong M_2|_V$ *for all* $V \in \mathfrak{Z}\mathfrak{U}$.

REMARK 1. One can also prove Corollary 3 more directly with an induction argument, using Brauer's theory of modular characters ($\mathfrak{U} = \{E\}$) for the start of the induction and Green's Transfer theorem (see [**5**]) for the induction step.

REMARK 2. Corollary 3, (b) implies easily Theorem 3.8 in [**7**].

REMARK 3. If $V \in \mathfrak{Z}\mathfrak{U}$, one can find two $\mathfrak{U}$-projective KG-modules M_1, M_2 with $M_1|_H \cong M_2|_H$ for any $H \leqq G$, which does not contain any conjugate of V, but $M_1|_V \ncong M_2|_V$ and a fortiori $M_1 \ncong M_2$. Therefore—up to conjugation—the maximal elements in $\mathfrak{Z}\mathfrak{U}$ is the smallest (i.e. best) possible family of subgroups in G, for which Corollary 3—and thus also Theorem 2—can hold.

REMARK 4. The image of $a(G, \mathfrak{U})$ in $\prod_{V \in \mathfrak{Z}\mathfrak{U}} a(V)$ is exactly the image of $a(G)$ in $\sum_{V \in \mathfrak{Z}\mathfrak{U}} a(V)$ and this is to some extent described in [2]. It seems that in this rather general situation not much more can be done. For more precise information in some special cases see [7], [8].

Theorems 1 and 2 are easy consequences of the following

THEOREM 3. *With* $\mathfrak{H}\mathfrak{U}=\{V\leqq G \mid V_p \trianglelefteq G,\ V/V_p$ *hyperelementary,* $V_p\subseteq U$ *for some* $U\in\mathfrak{U}\}$ *there exist a* p*-power* p^n *and integral numbers* $n_V(V\in\mathfrak{H}\mathfrak{U})$ *with* $p^n\cdot 1=\sum_{V\in\mathfrak{H}\mathfrak{U}} n_V\cdot K[G/V]$ *in* $a(G,\mathfrak{U})$ *with* $K[G/V]$ *the KG-module spanned by the cosets* G/V.

Using Corollary 3 for $\mathfrak{U}=\{G\}$ Theorem 3 reduces itself easily to the case $G_p\trianglelefteq G$, G_p elementary abelian and $\mathfrak{U}=\mathfrak{M}(G_p)$, the family of maximal subgroups in G_p. In this special case one can explictly construct the numbers p^n, $n_V(V\in\mathfrak{H}\mathfrak{U})$, using constructions based on Lemma 1.

Theorem 3 implies also a statement similar to Corollary 3: For any $\mathfrak{U}$-projective KG-module M there exists a p-power p^n and KV-modules $M_1(V)$, $M_2(V)$ $(V\in\mathfrak{H}\mathfrak{U})$ with

$$\underbrace{M\oplus\cdots\oplus M}_{(p^n\ \textit{times})}\oplus\bigoplus_{V\in\mathfrak{H}\mathfrak{U}} M_1(V)^{V\to G}\cong\bigoplus_{V\in\mathfrak{H}\mathfrak{U}} M_2(V)^{V\to G}.$$

One may also assume either $M_1(V)=0$ or $M_2(V)=0$ for any $V\in\mathfrak{H}\mathfrak{U}$ (the same remark holds for Corollary 3).

REMARK 5. If R is any commutative ring, one may as well form the relative Grothendieck ring $a(G,\mathfrak{U};R)$, spanned by the isomorphism classes of finitely generated RG-modules, modulo the span of the Euler characteristics of $\mathfrak{U}$-split exact sequences of RG-modules. One has also the notion of $\mathfrak{U}$-projective RG-modules [3] and a Cartan map κ: $k(G,\mathfrak{U};R)\to a(G,\mathfrak{U};R)$. One can still prove that the kernel of κ is a torsion submodule of $k(G,\mathfrak{U};R)$, probably it is even injective, but the cokernel is no longer a torsion module. (Counterexample: G cyclic of order p, $\mathfrak{U}=\{E\}$, $R=\hat{\boldsymbol{Z}}_p$.) But the following form of Theorem 2 may still be true:

CONJECTURE. With $\mathfrak{C}_R\mathfrak{U}=\{V\leq G \mid \exists N\trianglelefteq V,\ V/N$ cyclic, $N\subseteq U$ for some $U\in\mathfrak{U}$ and N a p-group for some p with $pR\neq R$, i.e. p not a unit in $R\}$ the map

$$a(G,\mathfrak{U};R)\to\prod_{V\in\mathfrak{C}_R\mathfrak{U}} a(V,V\cap\mathfrak{U};R)$$

has torsion kernel with

$$V\cap\mathfrak{U}=\{V\cap gUg^{-1}\mid g\in G,\ U\in\mathfrak{U}\}.$$

For $\mathfrak{U}=\{E\}$ this is a Corollary to the results of Swan [9], for $\mathfrak{U}=\{G\}$ this is the principal result in [2].

REMARK 6. There is another perhaps interesting application of the construction, used in the proof of Lemma 1: Let R be a commutative noetherian ring, whose maximal ideal spectrum is of dimension $\leqq n$. Let G be a finite group and consider the Grothendieck ring $a(G; R) = a(G, \{G\}; R)$ in the above notation. For any maximal ideal $\mathfrak{m}$ one has a homomorphism $a(G; R) \to a(G, R_{\mathfrak{m}})$ with kernel say $i(G; R, \mathfrak{m})$. Then $\bigcap_{\mathfrak{m}} i(G; R, \mathfrak{m}) = i(G, R)$ (where $\mathfrak{m}$ runs through all maximal ideals in R) is nilpotent of order $n+1$, i.e. $i(G, R)^{n+1}=0$.

Especially if R is a Dedekind ring in an algebraic number field, $i(G, R)$ is known to be the torsion submodule $\operatorname{Tor}(a(G; R))$ of $a(G; R)$. Thus $x \cdot y = 0$ in $a(G; R)$ for any two torsion elements $x, y \in a(G; R)$.

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