A NOTE ON FUNCTORS Ext OVER THE RING Z[1]

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Let A and B be modules over the ring Z of all integers. In this paper, we shall define a new homomorphism

$$\Gamma:\ B \otimes_Z \mathrm{Hom}_Z(A,\ Q/Z) \to \mathrm{Ext}_Z^1(A,\ B)$$

by $\Gamma(b \otimes h) = bE_0h$, for each $b \otimes h \in B \otimes_Z \mathrm{Hom}_Z(A,\ Q/Z)$ and check the properties of Γ, where E_0: $0 \to Z \to Q \to Q/Z \to 0$ is the familiar exact sequence and Q is the field of all rational numbers.

For convenience, in sequel we shall use $\otimes$, Hom and Ext for $\otimes_Z$, Hom_Z and Ext_Z^1, respectively, and A, B as Z-modules.

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The detailed definition of Γ is described by the diagram with each commutative square

$$\begin{array}{llcccccl}
bE_0h: & 0 \to B & \longrightarrow & D_2 & \longrightarrow & A & \longrightarrow & 0(\text{exact}) \\
 & \ \ b\uparrow & \text{push-out} & \uparrow & & \| & & \\
 & 0 \to Z & \longrightarrow & D_1 & \longrightarrow & A & \longrightarrow & 0(\text{exact}) \\
 & \ \ \ \| & & \downarrow & \text{pull-back} & \downarrow h & & \\
E_0: & 0 \to Z & \longrightarrow & Q & \longrightarrow & Q/Z & \longrightarrow & 0(\text{exact})
\end{array}$$

for each $b \otimes h \in B \otimes \mathrm{Hom}(A,\ Q/Z)$, where $b \in B$ is a homomorphism from Z to B such that $b(1) = b$.

By the standard methods as in [3] we know that for b_i $(i=0,\ 1,\ 2)$ in B and h_i $(i=0,\ 1,\ 2)$ in $\mathrm{Hom}(A,\ Q/Z)$ $(b_1+b_2)E_0h_0 = b_1E_0h_0 + b_2E_0h_0$, $b_0E_0(h_1+h_2) = b_0E_0h_1 + b_0E_0h_2$. Furthermore, for each f: $A_2 \to A_1$ and g: $B_1 \to B_2$, where A_i $(i=1,\ 2)$ and B_i $(i=1,\ 2)$ are Z-modules, we get the Z-homomorphisms

$$f_H^*:\ \mathrm{Hom}(A_1,\ Q/Z) \to \mathrm{Hom}(A_2,\ Q/Z), \qquad f_E^*:\ \mathrm{Ext}(A_1,\ B) \to \mathrm{Ext}(A_2,\ B)$$

$$g_E^*:\ \mathrm{Ext}(A,\ B_1) \to \mathrm{Ext}(A,\ B_2)$$

and in this case we also know that for each $b \otimes h_1 \in B \otimes \mathrm{Hom}(A_1,\ Q/Z)$ and $b_1 \otimes h \in B_1 \otimes \mathrm{Hom}(A,\ Q/Z)$

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$$f_B^* \cdot \Gamma(b \otimes h_1) = bE_0(h_1 f) = \Gamma \cdot 1_B \otimes f_H^*(b \otimes h_1),$$

$$g_B^* \cdot \Gamma(b_1 \otimes h) = g(b_1)E_0 h = \Gamma \cdot g \otimes 1_H(b_1 \otimes h)$$

where 1_H: Hom$(A, Q/Z) \to$ Hom$(A, Q/Z)$ is the identity map. This description implies that Γ *is natural in each argument.*

In general Γ is not an isomorphism because if we take $A = Z$ then Ext$(A, B) =$ Ext$(Z, A) = 0$ and $B \otimes$Hom$(A, Q/Z) = B \otimes$Hom$(Z, Q/Z)$ $\neq 0$ when B is not divisible. As a special case the following holds.

THEOREM 1. *If A is a cyclic Z-module $Z_m(a)$ of order m with generator a then Γ is an isomorphism.*

PROOF. To prove this theorem we should define an isomorphism

$$\eta: B \otimes \mathrm{Hom}(Z_m(a), Q/Z) \to \mathrm{Ext}(A, B)$$

by the following way.

First step. Define $\eta_1: B \otimes$Hom$(Z_m(a), Q/Z) \to B/mB$ by $\eta_1(b \otimes r)$ $= \eta_1(rb \otimes \alpha_m) = rb + mB$ for each $b \otimes r \in B \otimes$Hom$(Z_m(a), Q/Z)$, where $mB = \{mb \mid b \in B\}$ and

$$r: Z_m(a) \to Q/Z \quad \text{such that} \quad r(a) = \frac{r}{m},$$

$$\alpha_m: Z_m(a) \to Q/Z \quad \text{such that} \quad \alpha_m(a) = \frac{1}{m}.$$

If we define $\eta_1^{-1}: B/mB \to B \otimes$Hom$(Z_m(a), Q/Z)$ by $\eta_1^{-1}(b + mB)$ $= b \otimes \alpha_m$ for each $b + mB \in B/mB$ then we know that $\eta_1^{-1}\eta_1 =$ the identity map in $B \otimes$Hom$(A, Q/Z)$, $\eta_1\eta^{-1} =$ the identity map in B/mB which implies that η_1 is an isomorphism.

Second step. Define $\eta_2: B/mB \to$ Ext$(Z_m(a), B)$ by $\eta_2(b + mB) = E_b$ for each $b + mB \in B/mB$, where E_b: $0 \to B \xrightarrow{\kappa} E_b \xrightarrow{\sigma} Z_m(\mathrm{a}) \to 0$ (exact) such that for $\sigma(u) = a(u \in E_b)\kappa(b) = mu$. Then η_2 is an isomorphism (see Proposition 1.1 on p. 64 of [3]).

Third step. We shall define $\eta = \eta_2\eta_1$ by $\eta(b \otimes r) = \eta(rb \otimes \alpha_m) = E_{rb}$ for each $b \otimes r \in B \otimes$Hom$(Z_m(a), Q/Z)$, i.e.,

$$\eta = \eta_2\eta_1: B \otimes \mathrm{Hom}(Z_m(a), Q/Z) \to B/mB \to \mathrm{Ext}(Z_m(a), B)$$

$$b \otimes r = rb \otimes \alpha_m \mapsto rb + mB \mapsto E_{rb}$$

then η is an isomorphism.

Using η we shall verify our theorem. To do so, we have to prove that $\Gamma = \eta$ by showing that $E_{rb} = bE_0 r$. By our definitions we get that

$$\begin{array}{lccccccc} E_{rb}\colon & 0 \to B & \to & E_{rb} & \to & Z_m(a) & \to 0 \text{ (exact)} \\ & & & \Cup & & \Cup & \\ & \Cup & & u & \mapsto & a & \\ & rb_1 & \mapsto & mu & \mapsto & 0 & \end{array}$$

and

$$\begin{array}{lccccc} bE_0r\colon & 0 \to B & \longrightarrow & D_2 & \longrightarrow & Z_m(a) \to 0 \text{ (exact)} \\ & & & \Cup & & \Cup \\ & & & \left(0, \left(\dfrac{r}{m}, a\right)\right) & \mapsto & a \\ & & & (0, (r, 0)) & & \\ & \Cup & & \| & & \\ & rb & \mapsto & (rb, 0) & \to & 0 \end{array}$$

because $(0, (r, 0)) = (rb, 0)$ in D_2 (see the first part of this paper). Therefore $E_{rb} = bE_0r$ for each $b \otimes r \in B \otimes \mathrm{Hom}(A, Q/Z)$, which means $\eta = \Gamma$, and we complete our proof.

Let A be finite then A is a direct sum of a finite number of cyclic Z-modules, i.e., $A = \sum_{i=1}^{n} Z_{m_i}$($Z_{m_i}$: cyclic Z-module of order m_i). Using Theorem 1 the following is easily proved.

COROLLARY 1. *With the above situation*

$$\Gamma\colon\ B \otimes \mathrm{Hom}\left(\sum_{i=1}^{n} Z_{m_i}, Q/Z\right) \to \mathrm{Ext}\left(\sum_{i=1}^{n} Z_{m_i}, B\right)$$

is an isomorphism. (*Note*: $\mathrm{Hom}(A \oplus B, C) = \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C)$ *and* $\mathrm{Ext}(A \oplus B, C) = \mathrm{Ext}(A, C) \oplus \mathrm{Ext}(B, C)$.)

COROLLARY 2. *If* $A = \sum_\alpha A_\alpha$ *(direct sum) and* B *is finitely generated then* $\Gamma\colon B \otimes \mathrm{Hom}(A, Q/Z) \to \mathrm{Ext}(A, B)$ *is an isomorphism, where each* A_α *is finite.*

PROOF. We know that

$$\mathrm{Hom}\left(\sum_\alpha A_\alpha, Q/Z\right) \cong \prod_\alpha \mathrm{Hom}(A_\alpha, Q/Z),$$

$$\mathrm{Ext}\left(\sum_\alpha A_\alpha, B\right) \cong \prod_\alpha \mathrm{Ext}(A_\alpha, B)$$

(see pp. 97–98 of [1]) and

$$B \otimes \prod_{\alpha} \operatorname{Hom}(A, Q/Z) \cong \prod_{\alpha} (B \otimes \operatorname{Hom}(A_\alpha, Q/Z))$$

because B is finitely generated (see p. 32 of [1]). By Corollary 1 for each α $\Gamma\alpha: B\otimes \operatorname{Hom}(A_\alpha, Q/Z) \to \operatorname{Ext}(A_\alpha, B)$ is an isomorphism and therefore Γ is also an isomorphism.

Next, we shall consider the case which A is an infinite torsion module, i.e., $A = \operatorname{inj}\lim_\alpha A_\alpha(A_\alpha$: finite). In this case, in general

$$\Gamma: \quad B \otimes \operatorname{Hom}(A, Q/Z) \to \operatorname{Ext}(A, B)$$

is not an isomorphism as in the following example.

EXAMPLE. Set $B = Q$ and $A = Q/Z$. Since Q/Z is divisible $\operatorname{Hom}(Q/Z, Q/Z)$ is torsion-free (see Corollary 1.5 on p. 128 of [1]). Therefore $Q\otimes\operatorname{Hom}(Q/Z, Q/Z) \neq 0$. On the other hand $\operatorname{Ext}(Q/Z, Q) = 0$ (Q is injective). This shows that Γ is not an isomorphism. But the following holds.

THEOREM 2. *If A is an infinite torsion module and B is finitely generated then Γ is an isomorphism.*

PROOF. Put $A = \operatorname{inj}\lim_\alpha A_\alpha(A_\alpha$: finite). Let us assume $\phi_{\alpha'\alpha}: A_\alpha \to A_{\alpha'}$ for $\alpha < \alpha'$ and $\phi_\alpha: A_\alpha \to A$ (injection) such that $\phi_{\alpha'}\phi_{\alpha'\alpha} = \phi_\alpha$. We then have the commutative diagram

$$\begin{array}{ccc} & B\otimes\operatorname{Hom}(A, Q/Z) \cong B\otimes\operatorname{proj}\lim_\alpha \operatorname{Hom}(A_\alpha, Q/Z) & \\ {\scriptstyle 1_B\otimes\operatorname{Hom}(\phi_\alpha, Q/Z)} \swarrow & & \searrow {\scriptstyle 1_B\otimes\operatorname{Hom}(\phi_{\alpha'}, Q/Z)} \\ B\otimes\operatorname{Hom}(A_\alpha, Q/Z) & \xleftarrow{\;1_B\otimes\operatorname{Hom}(\phi_{\alpha'\alpha}, Q/Z)\;} & B\otimes\operatorname{Hom}(A_{\alpha'}, Q/Z). \end{array}$$

Therefore there exists a unique homomorphism θ as in the diagram with each triangle commutative

$$\begin{array}{ccccc} & & B\otimes\operatorname{Hom}(A, Q/Z) \cong B\otimes\operatorname{proj}\lim \operatorname{Hom}(A_\alpha, Q/Z) & & \\ & {\scriptstyle 1_B\otimes\operatorname{Hom}(\phi_\alpha, Q/Z)} \swarrow & \downarrow {\scriptstyle \exists!\theta} & \searrow {\scriptstyle 1_B\otimes\operatorname{Hom}(\phi_{\alpha'}, Q/Z)} & \\ B\otimes\operatorname{Hom}(A_\alpha, Q/Z) & \xleftarrow{\phi'_\alpha} & \operatorname{proj}\lim_\alpha(B\otimes\operatorname{Hom}(A_\alpha, Q/Z)) & \xrightarrow{\phi'_{\alpha'}} & B\otimes\operatorname{Hom}(A_{\alpha'}, Q/Z), \end{array}$$

where ϕ'_α and $\phi'_{\alpha'}$ are projections. On the other hand, since

$$\Gamma_\alpha: \quad B \otimes \operatorname{Hom}(A_\alpha, Q/Z) \to \operatorname{Ext}(A_\alpha, B),$$

for each α, is an isomorphism by Corollary 1 we have the isomorphism $\operatorname{proj}\lim_\alpha \Gamma_\alpha$: $\operatorname{proj}\lim_\alpha(B\otimes\operatorname{Hom}(A, Q/Z) \cong \operatorname{proj}\lim_\alpha \operatorname{Ext}(A_\alpha, B)$. Therefore, by the definition of the inverse limits and the naturality of Γ we have two commutative diagrams

$$\begin{array}{ccccc} & & \operatorname{Ext}(A, B) & & \\ & \swarrow^{\operatorname{Ext}(\phi_\alpha, B)} & \downarrow & \searrow^{\operatorname{Ext}(\phi_{\alpha'}, B)} & \\ \operatorname{Ext}(A_\alpha, B) & \xleftarrow{\phi_\alpha''} & \operatorname{proj\,lim}_\alpha(B \otimes \operatorname{Hom}(A_\alpha, Q/Z)) & \xrightarrow{\phi_{\alpha'}''} & \operatorname{Ext}(A_{\alpha'}, B) \end{array}$$

and

$$\begin{array}{ccccc} B \otimes \operatorname{Hom}(A_\alpha, B) & \longleftarrow & \operatorname{proj\,lim}_\alpha(B \otimes \operatorname{Hom}(A_\alpha, Q/Z)) & \longrightarrow & B \otimes \operatorname{Hom}(A_\alpha, Q/Z) \\ \cong \downarrow \Gamma_\alpha & & \cong \downarrow \operatorname{proj\,lim}_\alpha \Gamma_\alpha & & \cong \downarrow \Gamma_{\alpha'} \\ \operatorname{Ext}(A_\alpha, B) & \longleftarrow & \operatorname{proj\,lim}_\alpha \operatorname{Ext}(A_\alpha, Q/Z) & \xrightarrow{\phi_{\alpha'}''} & \operatorname{Ext}(A_\alpha, Q/Z), \end{array}$$

where ϕ_α'' and $\phi_{\alpha'}''$ are projections.

Moreover, by the naturality of Γ the diagram

$$\begin{array}{ccc} B \otimes \operatorname{Hom}(A_\alpha, Q/Z) & \xleftarrow{1_B \otimes \operatorname{Hom}(\phi_\alpha', Q/Z)} & B \otimes \operatorname{Hom}(A, Q/Z) \\ \Gamma_\alpha \downarrow & & \downarrow \Gamma \\ \operatorname{Ext}(A_\alpha, B) & \xleftarrow{\operatorname{Ext}(\phi_\alpha, B)} & \operatorname{Ext}(A, B) \end{array}$$

is commutative. We then have the commutative diagram:

$$\begin{array}{ccc} B \otimes \operatorname{Hom}(A, Q/Z) & \xrightarrow{\Gamma} & \operatorname{Ext}(A, B) \\ \exists!\, \operatorname{proj\,lim}_\alpha \Gamma_\alpha \cdot \theta \searrow & & \swarrow \exists!\xi \\ & \operatorname{proj\,lim}_\alpha \operatorname{Ext}(A_\alpha, B) & \end{array}$$

By our hypothesis $\xi: \operatorname{Ext}(A, B) \cong \operatorname{proj\,lim}_\alpha \operatorname{Ext}(A_\alpha, B)$ (see page 793 of [2]) and $\operatorname{proj\,lim}_\alpha \Gamma_\alpha \cdot \theta: B \otimes \operatorname{Hom}(A, Q/Z) = \operatorname{proj\,lim}_\alpha \operatorname{Ext}(A_\alpha, B)$ (Note: $\operatorname{proj\,lim}_\alpha(B \otimes \operatorname{Hom}(A_\alpha, Q/Z)) \cong \operatorname{proj\,lim}_\alpha \operatorname{Ext}(A_\alpha, B)$ and B is finitely generated (see p. 32 of [1]). Therefore Γ is an isomorphism, as asserted.

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