ON SINGULAR INTEGRALS

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The note is divided into three sections. The first section is devoted to singular kernels in R^n. Most of the results of the section remain valid after some modifications, if we replace R^n by a locally compact group and the Lebesgue measure by the Haar measure of the group; the second section deals with those extensions.

In the third section we apply the results of the first section to obtain L^p estimates for kernels whose homogeneity is given over a one parameter group. These kernels have been first considered by M. de Guzman [2]; particular cases of these kernels are those studied by A. P. Calderón and A. Zygmund in [1]; and by E. B. Fabes and N. Rivière in [3].

1. Singular kernels. Let $\{U_\alpha, \alpha>0\}$ be a family of open subsets of R^n, satisfying:

(a) $0\in U_\alpha$; for $\alpha<\beta$, $U_\alpha\subset U_\beta$; $\bigcap_\alpha U_\alpha=\{0\}$, the closure of U_α compact.

(b) There exists $\phi(\alpha)$ continuously mapping R_+ onto R_+ such that

$$U_\alpha - U_\alpha \subset U_{\phi(\alpha)} \quad \text{and} \quad m(U_{\phi(\alpha)}) \leqq Am(U_\alpha)$$
$$U_\alpha - U_\alpha = \{z; z = x - y, x \in U_\alpha, y \in U_\alpha\}.$$

(Clearly $\alpha<\phi(\alpha)$), $m(\cdot)$ denotes the Lebesgue measure.

(c) The function $f(\alpha)=m(U_\alpha)$ is left continuous and $f(\alpha)\to\infty$ as $\alpha\to\infty$.

We shall say that the operator T defined over a class of measurable functions is sublinear if

$$|T(f+g)| \leqq |T(f)| + |T(g)|,$$
$$L^p(R^n) = \left\{f; \|f\|_p = \left(\int_{R^n} |f(x)|^p dx\right)^{1/p} < \infty\right\}.$$

THEOREM 1 (WEAK TYPE). *Let $\{U_\alpha\}$ be a family as above, T a sublinear operator defined in $L^1(R^n)\oplus L^p(R^n)$ satisfying:*

(i) *For $f\in L^p(R^n)\cap L^\infty(R^n)$, $|Tf(x)|\leqq|T_1f(x)|+|T_2f(x)|$ where $m\{x; |T_1f(x)|>t\}\leqq(c/t^p)\int_{R^n}|f|^p dx$ and $\|T_2f\|_{L^\infty}\leqq\|f\|_{L^\infty}$.*

(ii) *If $f\in L^1(R^n)$ with support contained in $x+U_\alpha$, and if*

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$$\int_{x+U_\alpha} f(y)dy = 0,$$

then there exists $W_{\alpha,x}$ *such that* $m(W_{\alpha,x}) \leqq Cm(U_\alpha)$ and

$$\int_{(W_{\alpha,x})'} |Tf(y)|\,dy \leqq C\int_{R^n} |f(y)|\,dy.$$

(W' denotes the complement of W.) Under these hypotheses, for $f \in L^q(R^n)$, $1 \leqq q \leqq p$

$$m(\{x, |Tf(x)| \geqq t\}) \leqq \frac{BC}{t^q}\int_{R^n} |f|^q dx$$

where B is an absolute constant depending only on the dimension and the constant A of the family $\{U_\alpha\}$. (Note. $L^\infty(R^n)$ can be replaced by $L^r(R^n)$, $r \geqq p$.)

DEFINITION. Let $k(x)$ be a measurable function. We shall say that $k(x)$ is a singular kernel for the family $\{U_\alpha\}$ iff:

(1) $k \in L^1(\Omega)$, for every bounded open subset of $R^n - \{0\}$ and $|\int_{U'_\alpha \cap U_\gamma} k(x)dx| \leqq C < \infty$ independently of α and γ. Moreover the limit of the integral exists as $\alpha \to 0$.

(2) For $\phi(\alpha)$ as in (b)

$$\int_{U_{\phi(\alpha)} \cap U'_\alpha} |k(x)|\,dx < C \qquad \text{(ind. of } \alpha\text{)}.$$

(3) For $x \in R^n$ set $h(x, \alpha) = \inf\{1/|t|\,;\, tx \in U_\alpha\}$ then

$$\int_{U_\alpha} h(x, \alpha)\,|k(x)|\,dx < C \qquad \text{(ind. of } \alpha\text{)}.$$

Set

$$\begin{aligned} k_{\alpha,\gamma}(x) &= k(x) \quad \text{for } x \in U'_\alpha \cap U_\gamma, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and for $f \in L^p(R^n)$ $(1 \leqq p \leqq \infty)$

$$K_{\alpha,\gamma}(f)(x) = \int_{R^n} k_{\alpha,\gamma}(x-y)f(y)dy.$$

THEOREM 2. *Let $k(x)$ be a singular kernel such that for $y \in U_\beta$,*

$$\int_{(U_{\phi(\beta)})'} | k(x - y) - h(x) | \, dx \leqq C \quad \text{(ind. of } y)$$

then:

(i) *For* $f \in L^1(R^n)$, $m(\{x, \ |K_{\alpha,\gamma}(f)| \geqq t\}) \leqq BC/t\|f\|_1$. *Moreover* $\lim_{\alpha\to 0;\, \gamma\to\infty} K_{\alpha,\gamma}(f)(x) = K(f)(x)$ *exists pointwise almost everywhere.*

(ii) *For* $f \in L^p(R^n)$, $(1 < p < \infty)$, $\|K_{\alpha,\gamma}(f)\|_p \leqq B_p C\|f\|_p$ *(*B_p *depending on* p*). Moreover* $\lim_{\alpha\to 0;\, \gamma\to\infty} K_{\alpha,\gamma}(f)(x) = K(f)(x)$ *exists pointwise almost everywhere and in the* L^p*-sense.*

2. **Singular kernels on locally compact groups.** The results obtained in the previous section can be extended to locally compact groups, after suitable modifications. More explicitly we replace R^n by a locally compact group G and the Lebesgue measure $m(\cdot)$ by the Haar measure $h(\cdot)$ of the group. In the definition of the family $\{U_\alpha\}$ conditions (c) and (b) should be replaced by

(c′) $f(\alpha) = h(U_\alpha)$ is left continuous and $f(U_\alpha) \to h(G)$ as $\alpha \to \infty$,

(b′) $U_\alpha - U_\alpha \subset U_{\phi(\alpha)}$, $\phi(\alpha) > \alpha$, ϕ continuous and onto, and $h(U_{\phi(\alpha)}) \leqq A h(U_\alpha)$.

Under conditions (c′) and (b′) Theorem 1 remains valid in G.

Finally condition (3), of the definition of singular kernels, should be replaced by

(3′) for $f \in L^2(G)$, $\|K_{\alpha,\gamma}(f)\|_2 \leqq C\|f\|_2$ C ind. of α and γ.

With such extension Theorem 2, parts (i) and (ii), remain valid in G.

The results remain valid if we use vector valued functions over Banach spaces. For Theorem 1, $f \in L^p(G, B_1)$ and T is a mapping from $L^1(G, B_1) \oplus L^p(G, B_1)$ into $\mathfrak{M}(G, B_2)$ the measurable functions from G to the Banach space B_2 (G with its Haar measure). In Theorem 2, $k(x)$ for each x should be a mapping from a Banach space B_1 into a Banach space B_2 and the absolute values should be replaced by the appropriate norms of the operators.

3. **Homogeneous kernels.** Let $\pi: R_+ \to \mathfrak{L}(R^n, R^n)$ with the properties

(1) $\pi(\lambda\mu) = \pi(\lambda)\pi(\mu)$, π continuous, $\pi(1) = I$ (I, identity),

(2) for $0 < \lambda \leqq 1$ $\|\pi(\lambda)\| \leqq \lambda$ ($\|\cdot\|$ denotes the norm of $\mathfrak{L}(R^n, R^n)$ the space of linear transformations of R^n).

For simplicity $\pi(\lambda)$ will be denoted by T_λ. Let $F(x, r) = \|T_{r^{-1}}(x)\|$, then for $x \neq 0$ there exists a unique solution $r = r(x)$ of the equation $\|T_{r^{-1}}(x)\| = 1$. Moreover the function $r(x)$ defines a metric in R^n with the properties

(i) $r(T_\lambda(x)) = \lambda r(x)$,

(ii) $r(x) = 1$ if $\|x\| = 1$,

(iii) if $\|x\| \leqq 1$ then $r(x) \geqq \|x\|$. Iff $\|x\| \geqq 1$, then $r(x) \leqq \|x\|$.

Let $x' = T_{r^{-1}(x)}(x)$. Then $\|x'\| = 1$ and hence x' can be expressed by a coordinate system in S^{n-1}. More explicitly:

$$\begin{aligned} x_1' &= \cos\phi_1 \cdots \cos\phi_{n-1} \\ &\vdots \\ x_n' &= \sin\phi_1. \end{aligned}$$

In this way we define a change of coordinates of polar type $x \to (r, \phi_1, \cdots, \phi_{n-1})$. To compute the Jacobian of the transformation, we observe that T_λ can be written uniquely as $T_\lambda = e^{P \ln \lambda}$, where P is an $n \times n$ real matrix.

Then

$$\mathfrak{J}(x; r, \phi_1, \cdots, \phi_{n-1}) = r^{(\mathrm{tr}\, P)-1} \det(P(x'), \partial x'/\partial\phi_1, \cdots, \partial x'/\partial\phi_{n-1})$$

($\mathrm{tr}(P)$ denotes the trace of P). Set

$$H(\phi) = \det(P(x'), \partial x'/\partial\phi_1, \cdots, \partial x'/\partial\phi_{n-1}).$$

DEFINITION 2. A function $k(x)$ is a homogeneous kernel with respect to $\{T_\lambda\}$ iff

(1) $k(x)$ is defined in $R^n - \{0\}$, $k \in L^1(S^{n-1})$ and $\int_{S^{n-1}} k(x')H(\phi)d_\phi = 0$,

(2) for $\lambda > 0$, $k(T_\lambda(x)) = \lambda^{-\mathrm{tr}\, P} k(x)$.

A homogeneous kernel is a singular kernel for the family $U_\alpha = \{x, r(x) < \alpha\}$.

Define

$$\begin{aligned} k_{\epsilon,R}(x) &= k(x) \quad \text{for } \epsilon \leqq r(x) \leqq R, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and $K_{\epsilon,R}(f) = k_{\epsilon,R} * f$ ($f \in L^p(R^n)$, $1 \leqq p < \infty$). As a consequence of Theorem 2

THEOREM 3. *Let $k(x)$ be a homogeneous kernel satisfying*

$$\int_{\{x; r(x) \geqq 3r(y)\}} |k(x-y) - k(x)|\, dx \leqq C \quad (\textit{ind. of } y).$$

Then:

(i) *For* $f \in L^1(R^n)$; $m(\{x, |k_{\epsilon,R}(f)| \geqq t\}) \leqq (BC/t)\|f\|_1$. *The* $\lim_{\epsilon\to 0; R\to\infty} k_{\epsilon,R}(f)(x) = k(f)(x)$ *exists pointwise almost everywhere.*

(ii) *For* $f \in L^p(R^n)$, $1 < p < \infty$, $\|k_{\epsilon,R}(f)\|_p \leqq B_p C\|f\|_p$ (B_p *depending on* p), *and the* $\lim_{\epsilon\to 0; R\to\infty} k_{\epsilon,R}(f)(x) = k(f)(x)$ *exists pointwise almost everywhere and in the* L^p*-sense.*

Another consequence of Theorem 2 is the following multiplier theorem:

THEOREM 4. *Assume that*

$$\sum_{|\alpha|\leq N}\int_{\frac{1}{2}\leq r(x)\leq 2}|D^\alpha a_\lambda(x)|^2dx \leqq C$$

where $N>\operatorname{tr}(P)/2$, $a(x)\in L^\infty(R^n)$ *and* $a_\lambda(x)=a(T_\lambda(x))$.

For $f\in C_0^\infty(R^n)$ *set* $T_a(f)=\mathfrak{F}^{-1}(a\mathfrak{F}(f))$ ($\mathfrak{F}(\cdot)$ *denotes the Fourier transform*). *Then* $\|T_a(f)\|_p\leqq B_pC\|f\|_p$ *for all* p, $1<p<\infty$.

Note. Theorems 3 and 4 remain valid if we consider $k(x)$ and $a(x)$ as operators from a Hilbert space H_1 into a Hilbert space H_2 ($f\in L^p(R^n, H_1)$).

When $P=I$, the homogeneous kernels described above are exactly those studied by A. P. Calderón and A. Zygmund in [1]. When P is diagonal and the diagonal elements are greater than or equal to one we obtain the kernels with mixed homogeneities studied by E. B. Fabes and N. M. Rivière [3].

The proofs of these results will appear elsewhere.

REFERENCES

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