APPLICATIONS OF RADON-NIKODÝM THEOREMS TO MARTINGALES OF VECTOR VALUED FUNCTIONS

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The purpose of this note is to announce some new martingale convergence theorems derived as consequences of the Radon-Nikodým theorems, theorems for vector measures of Métivier [7] and Rieffel [8]. The results announced here contain theorems of Scalora [9], Chatterji [1] and [2], A. and C. Ionescu-Tulcea [5] and Métivier [7]. Throughout this note, unless explicitly mentioned otherwise, (Ω, Σ, μ) is a fixed finite measure space, $\mathfrak{X}$ is a Banach space, $L^p(\mu, \mathfrak{X})$, $(1 \leqq p < \infty)$ is the space of all strongly measurable $\mathfrak{X}$-valued (equivalence classes of) functions f on Ω satisfying $\|f\|_p = (\int_\Omega \|f\|^p d\mu)^{1/p} < \infty$. If T is a directed set, $\{B_\tau, \tau \in T\}$ is an increasing net of sub-σ-fields (i.e. $\tau_1 \leqq \tau_2$ implies $B_{\tau_1} \subset B_{\tau_2}$), then $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^p(\mu, \mathfrak{X})$ $(1 \leqq p < \infty)$ if $f_\tau \in L^p(\mu, \mathfrak{X})$, f_τ is measurable relative to B_τ, and $\int_E f_{\tau_2} d\mu = \int_E f_{\tau_1} d\mu$ if $\tau_2 \geqq \tau_1$ and $E \in B_{\tau_1}$.

1. A characterization of mean convergent martingales in $L^p(\mu, \mathfrak{X})$. This section is devoted to characterizing mean convergent martingales in $L^p(\mu, \mathfrak{X})$. Basic to this section is the following

DEFINITION. A martingale $\{f_\tau, B_\tau, \tau \in T\}$ in $L^p(\mu, \mathfrak{X})$ is said to have property WCD (CD) if for each $\epsilon > 0$ there exists a weakly (norm) compact convex subset $K_\epsilon \subset \mathfrak{X}$ such that for each $\delta > 0$ there is an index $\tau_0 \in T$ and $E_0 \in B_{\tau_0}$ with $\mu(\Omega - E_0) < \epsilon$ such that for $\tau \geqq \tau_0$

$$\int_E f_\tau d\mu \in \mu(E) K_\epsilon + \delta U$$

$E \subset E_0$, $E \in B_\tau$. Where $U = \{x \in \mathfrak{X}: \|x\| \leqq 1\}$.

The following theorem is believed to be the first theorem which characterizes mean convergent martingales in $L^p(\mu, \mathfrak{X})$.

THEOREM 1. *Let $\{f_\tau, B_\tau, \tau \in T\}$ be a martingale in $L^p(\mu, \mathfrak{X})$ $(1 \leqq p < \infty)$. The net $\{f_\tau, \tau \in T\}$ is convergent in the L^p norm if and only if*

(i) $\sup_{\tau \in T} \|f_\tau\|_p \leqq M < \infty$ *for some* M;

(ii) $\{f_\tau, \tau \in T\}$ *is terminally uniformly integrable; i.e. given an* $\epsilon > 0$, *there is an index* $\tau_0 \in T$ *and* $\delta > 0$ *such that* $\tau \geqq \tau_0$ *and* $\mu(E) < \delta$ *imply*

$$\int_E \|f_\tau\| d\mu < \epsilon,$$

and

(iii) $\{f_\tau, B_\tau, \tau\in T\}$ *has property* WCD (CD).

If $p>1$, (ii) *is implied by* (i) *and may be dropped.*

Note that along with (i) and (ii), either WCD or CD is necessary and sufficient also. Naturally WCD is weaker than CD, but CD is an interesting necessary condition nevertheless.

If $T=N$, the positive integers with the natural order, the following corollary which is a consequence of a theorem of A. and C. Ionescu-Tulcea [5] follows immediately from Theorem 1.

COROLLARY 2. *Let* $\{f_n, B_n\}$ *be a martingale satisfying* (i), (ii), *and* (iii) *of Theorem* 1. *If* f_∞ *is the* $L^1(\mu, \mathfrak{X})$*-limit of* $\{f_n\}$, *then* $\lim_{n\to\infty} f_n(\omega) = f_\infty(\omega)$ *a.e.* $[\mu]$, *strongly.*

The Chatterji-Scalora [1] and [9] theorem is readily seen to be a consequence of Theorem 1 with the aid of

PROPOSITION 3. *Let* $\mathfrak{X}$ *have the Radon-Nikodým property with respect to* μ; *i.e. every* μ*-continuous* $\mathfrak{X}$*-valued countably additive set function* F *of bounded variation on* Σ *admits the representation*

$$F(E) = \int_E f d\mu \qquad E\in\Sigma$$

some $f\in L^1(\mu, \mathfrak{X})$. *Then every martingale* $\{f_\tau, B_\tau, \tau\in T\}$ *in* $L^p(\mu, \mathfrak{X})$ $(1\leqq p<\infty)$ *with*

$$\sup_\tau \|f_\tau\|_p \leqq M < \infty$$

has both WCD *and* CD.

In particular, all reflexive and separable dual Banach spaces $\mathfrak{X}$ have the Radon-Nikodým property with respect to μ.

2. **$L^1(\mu, \mathfrak{X})$-bounded martingales.** In this section a generalization of a classical result of Doob and more recent results of Krickeberg and Chatterji [2, Theorem 2] is given.

THEOREM 4. *Let* $\{f_\tau, B_\tau, \tau\in T\}$ *be a martingale in* $L^1(\mu, \mathfrak{X})$ *with property* WCD *and such that*

$$\sup_\tau \|f_\tau\|_1 \leqq M < \infty$$

for some M. *Then there exists* $f\in L^1(\mu, \mathfrak{X})$ *such that* $\lim_\tau f_\tau = f$ *in* μ*-measure. If* $T=N$, *then* $\lim_n f_n(\omega) = f(\omega)$ *a.e.* $[\mu]$ *as well.*

In view of Proposition 3, Theorem 4 clearly subsumes the result of Chatterji [2, Theorem 2] on $L^1(\mu, \mathfrak{X})$-bounded martingales.

REMARKS. Certain of the above results are true in the context of Orlicz spaces $L^\Phi(\mu, \mathfrak{X})$ of vector valued functions. Proofs of these and other results will appear elsewhere.

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