A NOTE ON SLIT MAPPINGS

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1. **Introduction.** Recently the unitary properties of Grunsky's matrix have been studied by several authors. Milin [5] was apparently the first to observe these properties, and Pederson [6], unaware of Milin's work, rediscovered them independently later.

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be a regular univalent function in the unit circle. The function

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{n,k=0}^{\infty} d_{nk} z^n \zeta^k$$

is then regular in $|z| < 1$, $|\zeta| < 1$.

Grunsky's matrix $B = (b_{nk})$, $b_{nk} = (nk)^{1/2} d_{nk}$, $n, k = 1, 2, \cdots$ plays an important role in the theory of univalent functions; for example, simple proofs of the Bieberbach conjecture for $n = 4$ were arrived at through its properties [2], [3].

If $1/f(z) = 1/z + c_0 + c_1 z + \cdots$ maps $|z| < 1$ onto a domain D such that the area (in the Lebesgue sense) of the complementary of D is zero—then Grunsky's matrix is unitary [5, Theorem 1], [6, Theorem 2.2]. As Milin pointed out, the area of the complementary of D is zero if and only if $\sum_{n=1}^{\infty} n |c_n|^2 = 1$. Following Pederson, these functions $f(z)$ will be referred as "slit mappings."

2. **Properties of slit mappings.** We now prove the following

THEOREM. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is a slit mapping then*

$$\frac{1}{f(z)} = \frac{1}{z} + c_0 + c_1 z + \cdots$$

either is of the form $1/z + c_0 + c_1 z$, $|c_1| = 1$, *or there are infinitely many nonvanishing coefficients* c_k.

PROOF. The above theorem may also be formulated in the following way:

If $f(z)$ *is a slit mapping such that*

$$\frac{1}{f(z)} = \frac{1}{z} + c_0 + c_1 z + \cdots + c_n z^n, \qquad c_n \neq 0, \tag{1}$$

then $n = 1$ *and* $|c_1| = 1$.

Let

(2) $$P_n\left[\frac{1}{f(z)}\right] = F_n(z) = \frac{1}{z^n} + \sum_{k=1}^{\infty} c_{nk} z^k$$

be the nth Faber polynomial associated with $f(z)$. Then we have by [7]

(3) $$c_{nk} = -\, n d_{nk}, \qquad n, k = 1, 2, \cdots.$$

In terms of the coefficients $b_{nk} = (nk)^{1/2} d_{nk}$, we may write

(4) $$b_{nk} = -\,(k/n)^{1/2} c_{nk}, \qquad n, k = 1, 2, \cdots.$$

By the unitary properties of B, we have

(5) $$\sum_{n=1}^{\infty} b_{kn} \bar{b}_{jn} = 0, \qquad k \neq j.$$

From (4) and (5) it follows that

(6) $$\sum_{n=1}^{\infty} n c_{kn} \bar{c}_{jn} = 0, \qquad k \neq j.$$

For proof of our theorem we now assume, to the contrary, that there exist $l > 1$ such that

(7) $$1/f(z) = 1/z + c_0 + c_1 z + \cdots + c_l z^l,$$

where $c_l \neq 0$.
Subtitution of $k = 1, j = l^2$ in (6) yields

(8) $$\sum_{n=1}^{\infty} n c_{1n} \bar{c}_{l^2, n} = 0.$$

Since

$$P_1\left[\frac{1}{f(z)}\right] = \frac{1}{z} + \sum_{k=1}^{\infty} c_{1k} z^k = \frac{1}{f(z)} - c_0 = \frac{1}{z} + \sum_{k=1}^{l} c_k z^k$$

it follows that

(9) $$c_{1k} = c_k, \qquad k = 1, 2, \cdots, l, \quad c_{1k} = 0 \text{ for } k > l.$$

From (8) and (9) we obtain

(8′) $$\sum_{n=1}^{l} n c_{1n} \bar{c}_{l^2, n} = 0.$$

Since $P_n(x)$ is a polynomial of degree n in x, we have by (7), for any natural n,

$$P_n\left[\frac{1}{f(z)}\right] = \frac{1}{z^n} + \sum_{k=1}^{ln} c_{nk}z^k \tag{10}$$

$$c_{n,ln} = (c_{1l})^n = (c_l)^n, \; c_{nk} = 0 \qquad \text{for } k > ln. \tag{11}$$

From the definition of the coefficients d_{nk}, it is clear that $d_{nk} = d_{kn}$. Following Schiffer we deduce from (3)

$$kc_{nk} = nc_{kn}, \qquad n, k = 1, 2, \cdots. \tag{12}$$

(This identity was first proved by Grunsky [4] and Schur [8].) From (11) and (12), we have

$$c_{kn} = 0, \qquad k > ln. \tag{13}$$

Substituting $k = l^2$ in (13) we get

$$c_{l^2,n} = 0, \qquad n = 1, 2, \cdots, l-1. \tag{14}$$

Using (11) for $n = l$ and (12) for $k = l^2$, $n = l$ it follows that

$$l^2 c_{l,l^2} = lc_{l^2,l} = l^2(c_l)^l = l^2(c_{1l})^l \tag{15}$$

equations (14) and (15) now yield

$$\sum_{n=1}^{l} nc_{1n}\bar{c}_{l^2,n} = l^2 c_{1l}(\bar{c}_{1l})^l = l^2 c_l(\bar{c}_l)^l \neq 0. \tag{16}$$

Since this contradicts (8′) we have proved that if $f(z)$ is a slit mapping such that

$$\frac{1}{f(z)} = \frac{1}{z} + c_0 + c_1 z + \cdots + c_n z^n, \qquad c_n \neq 0$$

then necessarily $n = 1$. But it follows then, from the condition $\sum_{k=1}^{\infty} k|c_k|^2 = 1$, that $|c_1| = 1$, and the proof is complete.

REMARK 1. In [1] the author considered properties of slit mappings and proved the above theorem for some particular cases.

REMARK 2. The above theorem contains a result of Pederson [6, Theorem 2.3], as a special case.

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