TRIANGULATING NONSIMPLY CONNECTED MANIFOLDS

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Lashof and Rothenberg have recently announced the following

THEOREM. *Let M^n be a compact topological manifold with boundary N^{n-1}, with fundamental group satisfying condition S.*

(a) *If $H^4(M; Z_2) = H^3(N; Z_2) = 0$, and $n \geqq 6$, M admits a PL manifold structure.*

(b) *If N already has a PL structure, $H^4(M; Z_2) = H^3(N; Z_2) = 0$ and $n \geqq 5$, then M admits a PL manifold structure agreeing with the given one on the boundary.*

The condition S is that $\pi_1(M \times T^k)$ and $\pi_1(\partial M \times T^k)$ satisfy the necessary conditions for the splitting theorems to hold, where T^k is the k-torus. If $\pi_1(M)$ and $\pi_1(\partial M)$ are free abelian, then condition S is satisfied. The purpose of this note is to relax the condition on the fundamental group.

THEOREM 1. *Let M^n be a closed, orientable topological manifold of dimension $n \geqq 7$ with $H^4(M; Z_2) = 0$. Then M has a PL structure.*

PROOF. By [1] and [2] or by [3], the stable homeomorphism conjecture is true in these dimensions, so M has a stable structure. By [4], $\pi_1(M)$ is generated by imbedded one spheres with product neighborhoods. Let $f_i: S_i^1 \times D^{n-1} \to M$ be such imbeddings, $i = 1, 2, \cdots, k$. We may assume that the $f_i(S_i^1 \times D^{n-1})$'s are disjoint. Let $0 < \alpha < 1$ and $D_\alpha^{n-1} = \{x \in R^{n-1} \mid \|x\| \leqq \alpha\}$. Henceforth we ignore the f_i and consider $S_i^1 \times D_\alpha^{n-1} \subset S_i^1 \times D^{n-1} \subset M$.

Let

$$\overline{M} = M - \bigcup_{i=1}^{k} (\operatorname{int}(S_i^1 \times D_\alpha^{n-1}))$$

and

$$M' = \overline{M} \cup \bigcup_{i=1}^{k} (D_i^2 \times S^{n-2}).$$

That is, perform surgery on M to kill $\pi_1(M)$. Then $\pi_1(M') = 0$ and $H^4(M'; Z_2) = 0$. By Lashof and Rothenberg, M' has a PL structure. Let $V = M - \bigcup_{i=1}^{k} (S_i^1 \times D_\alpha^{n-1}) = M' - \bigcup_{i=1}^{k} (D_i^2 \times S^{n-2})$.

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Then V is an open subset of M', so V has a PL structure. Let ϵ_i be the end of V contained in $S_i^1 \times D^{n-1}$. Then $\pi_1(\epsilon_i) = Z$, so by Siebenman, ϵ_i has a connected PL manifold neighborhood N_i with $N_i \cong \partial N_i \times [0, 1)$. Let $W_i = N_i \cup \partial(S_i^1 \times D_\alpha^{n-1})$. Then W_i is an h-cobordism between ∂N_i and $\partial(S_i^1 \times D_\alpha^{n-1}) \cong S^1 \times S^{n-2}$, and hence W_i satisfies the hypothesis of the theorem of Lashof-Rothenberg, and we extend the triangulations on ∂N_i and $\partial(S_i^1 \times D_\alpha^{n-1})$ to all of W_i. Doing this for each $i = 1, 2, \cdots, k$, we get a triangulation for M, the triangulation induced from M' on $V - \bigcup N_i$, the triangulation of W_i, and the natural triangulation on $S_i^1 \times D_\alpha^{n-1}$.

THEOREM 2. *Let M^n be a compact orientable manifold with boundary N^{n-1} and suppose $H^4(M; Z_2) = H^3(N; Z_2) = 0$. Then any PL structure on N extends to a PL structure on M, provided $n \geqq 8$.*

PROOF. The proof is essentially the same as the proof of Theorem 1. As before $\pi_1(N)$ is generated by imbedded 1-spheres with product neighborhoods. Since N is a PL manifold, we may assume that the imbeddings f_i: $S_i^1 \times D^{n-2} \to N$ are piecewise linear. We use the maps $f_i | S_i^1 \times D^{n-2}$ to attach handles $D_i^2 \times D_\alpha^{n-2}$ to M. Call the resulting manifold M_1. Then ∂M_1 is a PL manifold with trivial fundamental group. We now perform surgeries on M_1 as in the proof of Theorem 1 to get M'. Then $\pi_1(M') = \pi_1(\partial M') = 0$, $H^4(M'; Z_2) = H^3(\partial M'; Z_2) = 0$, so by Lashof and Rothenberg, M' has a PL structure agreeing with the given PL structure on $\partial M'$. Just as in Theorem 1, M_1 then has a PL structure agreeing with the given PL structure of $\partial M_1 = \partial M'$.

Let $W = M - \bigcup(f_i(S_i^1 \times D_\alpha^{n-2}))$. Then W is an open subset of M_1, and so W is a PL manifold. Let ϵ_i be the end of W contained in a neighborhood of $f_i(S_i^1 \times D_\alpha^{n-2})$. Then ϵ_i is tame and $\pi_1(\epsilon_i) = Z$. By the relative Siebenman theorem, ϵ_i is collared. That is, there is a connected PL manifold neighborhood V_i of ϵ_i such that V_i is closed in W, frontier of V_i is compact submanifold of W, and $V_i \cong \partial V_i \times [0, 1)$, $V_i \cap \partial W \cong \partial(V_i \cap \partial W) \times [0, 1)$. Let $U_i = V_i \cup_{f_i} (S_i^1 \times D_\alpha^{n-2})$. Then U_i is a compact topological manifold with a PL triangulation on ∂U_i. (The triangulations on ∂V_i and ∂M agree on the $\partial V_i \cap \partial M_i$). Now $\pi_1(U_i) = Z$, $H^4(U_i; Z_2) = H^3(\partial U_i; Z_2) = 0$, so by Lashof and Rothenberg, the triangulation on ∂U_i extends to a PL triangulation of U_i. Doing this for each i we get a PL structure on M that agrees with the given PL structure on ∂M.

I have been told that R. C. Kirby and/or L. C. Siebenman have proved a stronger result independently and previously. Theorems 1 and 2 may still be of interest, however, in that the proofs remain valid without the condition on the cohomology groups, provided one can remove this condition in the theorem of Lashof-Rothenberg.

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