ON INJECTIVE BANACH SPACES AND THE SPACES $C(S)$

BY HASKELL P. ROSENTHAL[1]

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A Banach space is injective (resp. a $\mathcal{P}_1$ space) if every isomorphic (resp. isometric) imbedding of it in an arbitrary Banach space Y is the range of a bounded (resp. norm-one) linear projection defined on Y.

In §1 we study linear topological properties of injective Banach spaces and the spaces $C(S)$ themselves; in §2 we study their conjugate spaces. (Throughout, "S" denotes an arbitrary compact Hausdorff space.) For example, by applying a result of Gaifman [3], we obtain in §1 that there exists a $\mathcal{P}_1$ space which is not isomorphic to any conjugate Banach space. We also obtain there that S satisfies the countable chain condition (the C.C.C.) if and only if every weakly compact subset of $C(S)$ is separable. (S is said to satisfy the C.C.C. if every uncountable family of open subsets of S contains two distinct sets with nonempty intersection.) In §2 we classify up to isomorphism (linear homeomorphism) all the conjugate spaces (B^*, B^{**}, B^{***}, etc.) of the $\mathcal{P}_1$ spaces $B=L^\infty(\mu)$ for some finite measure μ, or $B=l^\infty(\Gamma)$ for some infinite set Γ. (The isomorphic classification of the spaces $L^\infty(\mu)$ for finite measures μ is given in [8].) We also determine in §2 the injective quotients of the above spaces B, and show that every injective Banach space of dimension the continuum, has its dual isomorphic to $(l^\infty)^*$. (Dimension of a Banach space Y (denoted dim Y) equals the minimum of the cardinalities of subsets of Y with dense linear span.)

We include some of the proofs; full details of these and other results will appear in [7].

1. We say that S carries a *strictly positive measure* if there exists a $\mu \in M(S)$ (the space of bounded Radon measures on S) such that $\mu(U)>0$ for all nonempty open $U\subset S$.

THEOREM 1.1. *Let S satisfy the C.C.C. and suppose that $C(S)$ is isomorphic to a conjugate Banach space. Then S carries a strictly positive measure.*

PROOF. The hypotheses and the Riesz representation theorem imply that there exists a closed subspace A of $M(S)$ such that $C(S)$ is isomorphic to A^* and A is weak* dense in $M(S)$ (identifying $M(S)$

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with $C(S)^*$). If there exists a positive $\mu \in M(S)$ such that every member of A is absolutely continuous with respect to μ, then S carries a strictly positive measure, namely μ. If there does not exist such a μ, then there exists an uncountable set Γ such that $l^1(\Gamma)$ is isomorphic to a complemented subspace of A, by the Lemma of §1 of [8]. Then $l^\infty(\Gamma)$ is isomorphic to a subspace of $C(S)$, but this is impossible in view of the assumption that S satisfies the C.C.C., as follows from the next proposition (cf., also Theorem 1.4(a) below).

LEMMA 1.2. *Let S satisfy the C.C.C., and suppose that $\mathfrak{F}$ is an uncountable family of open subsets of S. Then there exists an infinite sequence $F_1, F_2, \cdots$ of distinct members of $\mathfrak{F}$ with $\bigcap_{i=1}^\infty F_i \neq \emptyset$.*

PROOF. Assume that no nonempty open subset of S is contained in uncountably many members of $\mathfrak{F}$. Define $\mathfrak{F}_n$ to be the class of all sets of the form $F_1 \cap \cdots \cap F_n$, where $F_1, \cdots, F_n$ are n distinct members of $\mathfrak{F}$. For each $A \in \mathfrak{F}_2$, put $\mathfrak{F}_A = \{F \in \mathfrak{F}: F \supset A\}$; and put $\mathfrak{K} = \bigcup \{\mathfrak{F}_A: A \in \mathfrak{F}_2 \text{ and } A \neq \emptyset\}$. $\mathfrak{F} \sim \mathfrak{K}$ is a pairwise disjoint family of open sets; hence $\mathfrak{F} \sim \mathfrak{K}$ is at most countable since S satisfies the C.C.C.; thus $\mathfrak{K}$ is uncountable. The above assumption implies that $\mathfrak{F}_A$ is at most countable for all nonempty A in $\mathfrak{F}_2$; hence $\mathfrak{F}_2$ is uncountable. We then obtain by induction that $\mathfrak{F}_n$ is uncountable for all n. Now let G_n be the set of all points in S which are contained in at most n distinct members of $\mathfrak{F}$, put G_n^0 equal to the interior of G_n, and let $\mathcal{G}_n = \{F \in \mathfrak{F}: F \cap G_n^0 \neq \emptyset\}$. Our initial assumption and the argument above imply that $\mathcal{G}_n$ is countable for all n. Hence there exists a nonempty F in $\mathfrak{F}$ such that $F \cap \bigcup_{n=1}^\infty G_n^0 = \emptyset$. F is of the second category in S by the Baire-category theorem, and hence every point of F except those belonging to the first category set $\bigcup_{n=1}^\infty G_n \sim G_n^0$, belongs to infinitely many members of $\mathfrak{F}$. Q.E.D.

REMARK. A modification of the above argument shows that the conclusion of Lemma 1.2 holds if we replace the assumption that S satisfies the C.C.C., by the hypotheses that card $\mathfrak{F} = \mathfrak{m}$ and that S satisfies the $\mathfrak{m}$-chain condition (every disjoint family of open sets has cardinality less than $\mathfrak{m}$) in its statement. A consequence of this is that if S is Stonian and Γ is a set with $c_0(\Gamma)$ isomorphic to a subspace of $C(S)$, then $l^\infty(\Gamma)$ is isometric to a subspace of $C(S)$. (We say that S is Stonian if every open subset of S has open closure.)

It follows from a result of Gaifman [3] that there exists a Stonian space, henceforth denoted S_G, satisfying the C.C.C. but carrying no strictly positive measure. It is known that a space is $\mathcal{P}_1$ if and only if it is isometric to $C(S)$ for some Stonian S (cf. [2]). Thus Gaifman's result and Theorem 1.1 yield the immediate

COROLLARY 1.3. *There exists a $\mathcal{P}_1$ space which is not isomorphic to any conjugate Banach space.*

The next result yields some basic linear topological invariants of the spaces $C(S)$.

THEOREM 1.4. (a) *S satisfies the C.C.C. if and only if every weakly compact subset of $C(S)$ is separable (if and only if $C(S)$ contains no isomorph of $c_0(\Gamma)$ for any uncountable set Γ).*

(b) *S carries a strictly positive measure if and only if $C(S)^*$ contains a weakly compact total subset.*

(b) follows easily from the Lemma of §1 of [8] (a total set is one whose linear span is weak* dense).

PROOF OF (a). We assume that S satisfies the C.C.C., and let K be a weakly compact subset of $C(S)$; we now show that K is separable (this is the only nontrivial implication). Suppose K is not separable; then by the Krein-Smulian theorem, K_1 the closed-convex-circled hull of K, is also weakly compact and nonseparable. Then by Proposition 3.4 of [4], K_1 contains a subset homeomorphic in its weak topology to the one-point compactification of an uncountable set. It follows that we may choose a $\delta > 0$, and an uncountable set $\Gamma \subset K_1$, with $\|\gamma\| > \delta$ for all $\gamma \in \Gamma$, such that every sequence of distinct elements of Γ converges weakly to zero. For each $\gamma \in \Gamma$, let $U_\gamma = \{s \in S: |\gamma(s)| > \delta/2\}$. Lemma 1.2 implies that there exists an infinite sequence $\gamma_1, \gamma_2, \cdots$ of distinct elements of Γ with $\bigcap_{i=1}^\infty U_{\gamma_i}$ nonempty. Then $\gamma_i \not\to 0$ weakly, a contradiction. Q.E.D.

An immediate consequence of Theorem 1.4(a) and the results of [1] is

COROLLARY 1.5. *Let K be a weakly compact subset of a Banach space and suppose that K satisfies the C.C.C. Then K is separable.*

Another consequence of 1.4(a) is that for any finite measure μ, every weakly compact subset of $L^\infty(\mu)$ is separable. An alternate proof of this fact may be given by using the following result (cf. §2 of [8] for the relevant definitions).

PROPOSITION 1.6. *Let the Banach space B be weakly compactly generated and satisfy the Dunford-Pettis property. Then every weakly compact subset of B^* is separable.*

The final result of this paragraph has as a consequence that if B is an injective conjugate Banach space with B^* weak* separable, then B is isomorphic to l^∞ if it is of infinite dimension.

THEOREM 1.7. *Let B be an injective Banach space that is isomorphic* *to a conjugate Banach space. Then the following conditions are equivalent*:

1. *B is isomorphic to a subspace of $L^\infty(\mu)$ for some finite measure μ.*
2. *If Γ is an uncountable set, then $l^\infty(\Gamma)$ is not isomorphic to a subspace of B.*
3. *Every weakly compact subset of B is separable.*
4. *B^* contains a weakly compact total subset.*
5. *There exists a finite measure μ and a closed subspace A of $L^1(\mu)$ such that B is isomorphic to A^*.*

We note that the $\mathcal{P}_1$ space $C(S_G)$ of Corollary 1.3 above satisfies conditions 2 and 3 but fails conditions 1, 4, and 5.

2. Throughout this paragraph, "μ" denotes a finite measure, $\mu_{\mathfrak{m}}$ denotes Lebesgue product measure on the product of $\mathfrak{m}$ copies of $[0, 1]$, and $\mathfrak{c}$ denotes the cardinality of the continuum. We regard $L^\infty(\mu)$ as a Banach algebra; by a "subalgebra" we shall mean a "conjugation-closed subalgebra." Given an indexed family $\{X_\alpha\}_{\alpha\in I}$ of Banach spaces, we denote by $\sum_{\alpha\in I}\oplus X_\alpha$ the Banach space consisting of all functions $x=\{x_\alpha\}_{\alpha\in I}$ with $x_\alpha\in X_\alpha$ for all α and $\|x\|=\sum_{\alpha\in I}\|x_\alpha\|_{X_\alpha}<\infty$. If $X_\alpha=X$ for all α and card $I=\mathfrak{m}$, we denote $\sum_{\alpha\in I}\oplus X_\alpha$ by $\sum_{\mathfrak{m}}\oplus X$.

A special case of the first and main result of this paragraph is that $(l^\infty)^*$ and $(L^\infty(\mu_{\mathfrak{c}}))^*$ are isometric, and $L^\infty(\mu_{\mathfrak{c}})$ is algebraically isometric to a quotient algebra of l^∞. We obtain in [7] that l^∞ and $L^\infty(\mu_{\mathfrak{c}})$ are not isomorphic.

THEOREM 2.1. *Let $\mathfrak{m}$ be an infinite cardinal number. Let B denote one of the Banach algebras $L^\infty(\mu)$ for some homogeneous μ, $l^\infty(\Delta)$ for some set Δ, or $C(G^{\mathfrak{m}})$ where G denotes the closed unit interval with endpoints identified; suppose* dim $B=\mathfrak{m}$. *Then*

(a) *B^* is isomorphic to $\sum_{2^{\mathfrak{m}}}\oplus L^1(\mu_{\mathfrak{m}})$.*

(b) *B^{**} is isomorphic to $l^\infty(\Gamma)$ where Γ is a set of cardinality $2^{\mathfrak{m}}$.*

(c) *Let $\mathfrak{e}_{\mathfrak{m}}$ denote the set of infinite cardinal numbers less than or equal to $\mathfrak{m}$, and for each $\mathfrak{n}\in\mathfrak{e}_{\mathfrak{m}}$, let $\Lambda_{\mathfrak{n}}$ be a set of cardinality $2^{\mathfrak{m}}$, with $\Lambda_{\mathfrak{n}}$ disjoint from $\Lambda_{\mathfrak{n}'}$ for $\mathfrak{n}\neq\mathfrak{n}'$. Then B^* is linearly isometric to*

$$l^1(\Gamma)\oplus\sum_{\mathfrak{n}\in\mathfrak{e}_{\mathfrak{m}}}\sum_{\alpha\in\Lambda_{\mathfrak{n}}}\oplus(L^1(\mu_{\mathfrak{n}}))_\alpha$$

where Γ is a set of cardinality $2^{\mathfrak{m}}$.

(d) *$C(G^{\mathfrak{m}})$ is algebraically isometric to a subalgebra of B.*

(e) *If S is Stonian with* dim $C(S)\leq\mathfrak{m}$, *then $C(S)$ is algebraically isometric to a quotient algebra of B.*

(f) *If Y is an injective Banach space with* dim $Y\leq\mathfrak{m}$, *then Y is isomorphic to a quotient space of B.*

(a) and (b) isomorphically determine all of the conjugate spaces of B as above and hence also of $C[0, 1]$, for one can show that $C[0, 1]^*$ is isometric to $C(G^{\aleph_0})^*$.

REMARK. Let S_1 be Stonian and satisfy the C.C.C., and suppose $\dim C(S_1) = \mathfrak{m}$. A recent result of Robert Solovay (unpublished) shows that (d) holds if we put $B = C(S_1)$; our proof of Theorem 2.1 then shows that all of the properties (a) through (f) hold for B. In particular, the word "homogeneous" may be deleted from the statement of Theorem 2.1; a proof of this special case of Solovay's result is given in [7].

Our next result is a consequence of the proof of Theorem 2.1 and a result of Grothendieck (cf. Theorem 4.3 of [4]).

PROPOSITION 2.2. *Let K be a weakly compact subset of a Banach space such that* card $K = \mathfrak{c}$, *and such that K contains a perfect nonempty subset. Then $C(K)^*$ is isometric to $C[0, 1]^*$.*

It is easy to construct such K for which $C(K)$ is nonseparable.

Our final result yields a large class of Banach spaces with duals isomorphic to the dual of l^∞; its proof uses critically the results of [6] and an argument of Pełczyński's (the proof of Proposition 4 of [5]).

THEOREM 2.3. *Let the Banach space Y be isomorphic to a quotient space of l^∞ and a complemented subspace of $C(S)$ for some S. Then Y^* is isomorphic to $(l^\infty)^*$.*

The hypotheses apply to any injective Y with $\dim Y = \mathfrak{c}$.

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UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720