THE SPACE OF CONTINUOUS LINEAR OPERATORS AS A COMPLETION OF $E' \otimes F$

BY J. W. BRACE[1]

Communicated by Leonard Gillman, January 27, 1969

The linear space $\mathcal{L}(E,F)$ of all continuous linear maps of the Banach space E into a Banach space F is the completion of $E' \otimes F$ (all continuous linear maps with finite dimensional range) for a Hausdorff locally convex topology t. The topology t is the supremum of the strong operator topology [2, p. 475] and the topology of uniform convergence on the unit ball of E, F having the $\sigma(F,F')$ topology. The notation is patterned after [4]. In this context the result is stated in the following way.

THEOREM. $\mathcal{L}_t(E,F) = \widehat{E' \otimes_t F}$.

The topology t is not a tensor product topology in the sense of Grothendieck [3, p. 88].

In reference [1] we treat the general question of the approximation of a class H of operators by the operators $E' \otimes F$. The class H is taken to be the linear operators, the continuous operators, the completely continuous operators, the weakly compact operators, or the compact operators. Topologies are given on a larger space G of linear operators such that H is a subspace which is either the closure or the completion of $E' \otimes F$. The above theorem is one example. In this case it is possible to give the following proof without reference to the more general approach used in [1].

PROOF. The space $L(E,F)$ of all linear maps of E into F is the completion of $E' \otimes F$ for the strong operator topology. Now consider $E' \otimes F$ as a collection of operators that map the Banach space F' into the Banach space E'. The space $L(F',E')$ of all linear maps of F' into E' is again the completion of $E' \otimes F$ for the strong operator topology. These statements are true because a Cauchy net for the strong operator topology always converges to a linear operator (F and E' are both complete); and, every linear operator is the limit of such a Cauchy net because for every finite dimensional subspace of the normed space E and of the space F' with the $\sigma(F',F)$ topology there is a continuous projection onto the subspace.

[1] The author was partially supported by a grant from the National Science Foundation, NSF GP-7492.

Consider the natural injection of $L(E,F)$ and $L(F',E')$ into the space of linear forms defined on $E\otimes F'$. For each f in $L(E,F)$ we have the linear form which maps $x\otimes y'$ to $<f(x), y'>$. For a g in $L(F',E')$ the corresponding linear form maps $x\otimes y'$ to $<x, g(y')>$. On the linear forms obtained from $E'\otimes F$ the two strong operator topologies can be expressed as uniform convergence on families $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. The families are composed of subsets of $E\otimes F'$ with uniform convergence on members of $\mathfrak{A}_1$ giving the strong operator topology obtained from $L(E, F)$ and uniform convergence on members of $\mathfrak{A}_2$ giving the strong operator topology obtained from $L(F', E')$. The family $\mathfrak{A}_1$ is all $\sigma(E\otimes F', E'\otimes F)$-closed balanced convex hulls of sets of the form $C=\{x_i\otimes y': i=1, 2, \cdots, n \text{ and } \|y'\|\leqq 1\}$ where the set $(x_1, x_2, \cdots, x_n)$ ranges through all finite subsets of E. In a similar manner the members of $\mathfrak{A}_2$ are obtained from sets of the form $\{x\otimes y_j': j=1, \cdots, m \text{ and } \|x\|\leqq 1\}$. The members of $\mathfrak{A}_1$ are compact for the $\sigma(E\otimes F', E'\otimes F)$ topology because the sets of the form C are compact and the closed convex hull of each such set is a subset of the compact set $C+\cdots+C$ where the sum is taken n times [5, p. 35].

When we consider $L(E,F)$ as linear forms, Grothendieck's completion theorem [4, p. 248], [5, p. 145] tells us that $L(E,F)$ is all linear forms whose restrictions to members of $\mathfrak{A}_1$ are $\sigma(E\otimes F', E'\otimes F)$ continuous. By the same reasoning, $L(F',E')$ is all linear forms whose restrictions to members of $\mathfrak{A}_2$ are continuous. Now consider $\mathfrak{L}(E, F)$ as linear forms on $E\otimes F'$ and observe that $\mathfrak{L}(E, F)=L(E, F)\cap L(F', E')$. This is because $\mathfrak{L}(E, F)$ is all members of $L(E, F)$ which have adjoints defined on F', i.e. all linear maps which are weakly continuous and thus continuous [5, p. 199]. Thus the injective image of $\mathfrak{L}(E,F)$ is all linear forms whose restrictions to members of $\mathfrak{A}_1\cup\mathfrak{A}_2$ are $\sigma(E\otimes F', E'\otimes F)$ continuous.

Direct computation will verify that a linear form which has continuous restrictions on $A_1\in\mathfrak{A}_1$ and $A_2\in\mathfrak{A}_2$ has a continuous restriction on A_1+A_2, because of the compactness of A_1. It is also true that A_1+A_2 is closed and contains the convex hull of $A_1\cup A_2$. This tells us that all linear forms from the injection of $\mathfrak{L}(E,F)$ are precisely a linear forms whose restrictions to the closed balanced convex hulls of members of $\mathfrak{A}_1\cup\mathfrak{A}_2$ are $\sigma(E\otimes F', E'\otimes F)$ continuous.

A final application of Grothendieck's completion theorem results in the injection of $\mathfrak{L}(E,F)$ being the completion of $E'\otimes F$ for the topology of uniform convergence on members of $\mathfrak{A}_1\cup\mathfrak{A}_2$. We reverse the injection to obtain the desired result for the linear space $\mathfrak{L}(E,F)$ of continuous linear operators. The topology of uniform convergence on members of $\mathfrak{A}_1 \cup \mathfrak{A}_2$ becomes the supremum of the strong operator topology and the topology of uniform convergence on the unit ball of E with F having the $\sigma(F,F')$ topology. The latter topology was the strong operator topology on $L(F',E')$ before we went over to the linear forms.

References

1. J. W. Brace and P. J. Richetta, *The approximation of linear operators*, Trans. Amer. Math. Soc. (to appear).

2. N. Dunford and J. T. Schartz, *Linear operators*. Part I, Interscience, New York, 1958.

3. A. Grothendieck, *Produits tensoriels topologiques*, Mem. Amer. Math. Soc. No. 16, 1955.

4. J. Horvath, *Topological vector spaces and distributions*, Addison-Wesley, Reading, Mass., 1966.

5. J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, N. J., 1963.

University of Maryland, College Park, Maryland 20742