ON SIMULTANEOUS APPROXIMATION AND INTERPOLATION WHICH PRESERVES THE NORM

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In [6] H. Yamabe established the following "simultaneous approximation and interpolation" theorem, which generalized a result of Walsh [4, p. 310] (cf. also [1], [3] for further generalizations), and is related to a theorem of Helly in the theory of moments (cf. e.g. [2, pp. 86–87]).

THEOREM (YAMABE). *Let M be a dense convex subset of the real normed linear space X, and let $x_1^*, \cdots, x_n^* \in X^*$. Then for each $x \in X$ and each $\epsilon > 0$, there exists a $y \in M$ such that $\|x-y\| < \epsilon$ and $x_i^*(y) = x_i^*(x)$ $(i=1, \cdots, n)$.*

Wolibner [5], in essence, proved that Yamabe's theorem could be sharpened in the particular case when $X = C([a, b])$, $M = \mathcal{P} =$ "the polynomials," and the x_i^* are "point evaluations." Indeed, from the results of [5] there can readily be deduced the following

THEOREM (WOLIBNER). *Let $a \leqq t_1 < \cdots < t_n \leqq b$ and let $\mathcal{P}$ be the set of polynomials. Then for each $x \in C([a, b])$ and each $\epsilon > 0$, there exists a $p \in \mathcal{P}$ such that $\|x-p\| < \epsilon$, $p(t_i) = x(t_i)$ $(i=1, \cdots, n)$, and $\|p\| = \|x\|$.*

Motivated by Wolibner's theorem, we consider the following more general problem. Let M be a dense subspace of the real normed linear space X, and let $\{x_1^*, \cdots, x_n^*\}$ be a finite subset of the dual space X^*. The triple $(X, M, \{x_1^*, \cdots, x_n^*\})$ will be said to have *property SAIN* (simultaneous approximation and interpolation which is norm-preserving) provided that the following condition is satisfied:

For each $x \in X$ and each $\epsilon > 0$ there exists a $y \in M$ such that $\|x-y\| < \epsilon$, $x_i^*(y) = x_i^*(x)$ $(i=1, \cdots, n)$, and $\|y\| = \|x\|$.

In this note we shall outline some of the main results we have obtained regarding property SAIN. Detailed proofs and related matter will appear elsewhere.

THEOREM 1. *Let M be a dense subspace of the Hilbert space X and let $x_1^*, \cdots, x_n^* \in X^*$. Then $(X, M, \{x_1^*, \cdots, x_n^*\})$ has property SAIN if and only if each x_i^* attains its norm on the unit ball in M.*

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The necessity in Theorem 1 is valid in *any* reflexive Banach space X. Whether the sufficiency is also valid in any reflexive Banach space is an open question. Also, in the case when $n=1$, Theorem 1 is valid in any strictly convex reflexive Banach space.

Let T denote a compact Hausdorff space and $C(T)$ the real continuous functions on T with the sup norm. If $t\in T$, δ_t will denote the functional "point evaluation" at t, i.e. $\delta_t(x)=x(t)$ for all $x\in C(T)$.

THEOREM 2. *Let A be a dense subalgebra of $C(T)$ and $t_1, \cdots, t_n\in T$. Then $(C(T), A, \{\delta_{t_1}, \cdots, \delta_{t_n}\})$ has property SAIN.*

Theorem 2 contains that of Wolibner and represents a strengthening of the Stone-Weierstrass theorem. Theorem 2 is proved by a rather tedious induction on n using Yamabe's theorem and the following lemma which essentially allows us to approximate the unit function in a useful manner.

LEMMA. *Let A and t_i be as in Theorem 2. Then for each $\epsilon>0$, there exists an element $e\in A$ such that $\|e-1\|<\epsilon$, $e(t_i)=1$ $(i=1, \cdots, n)$, and $e\leqq 1$.*

Theorem 2 is also valid if "dense subalgebra" is replaced by "dense linear sublattice containing constants." However, the following examples show that these results cannot be extended very far.

EXAMPLE 1. Let

$$M = \{x \in C([0, 1]): x'(\tfrac{1}{2}) \text{ exists}, x'(\tfrac{1}{2}) = x(0) - x(1)\}.$$

Then M is a *dense subspace of* $C([0, 1])$, which contains constants, but such that $(C([0, 1]), M, \delta_{1/2})$ does *not* have property SAIN (since if $x\in C([0, 1])$ is the function which is 1 if $0\leqq t\leqq\frac{1}{2}$ and $x(t)=-2t+2$ if $\frac{1}{2}<t\leqq 1$, and y is any element of M which satisfies $y(\frac{1}{2})=x(\frac{1}{2})=1$ and $\|y\|=\|x\|=1$, then $y'(\frac{1}{2})=0$ so $y(0)=y(1)$ and hence $\|x-y\|\geqq\frac{1}{2}$).

EXAMPLE 2. Let $A=\text{span}\{x_1, x_2, \cdots\}$ where $x_i(t)=t^i$ $(i=1, 2, \cdots)$ and define x^* by $x^*(x)=\int_1^2 x(t)dt$ for all $x\in C([1, 2])$. Then A is a *dense subalgebra* of $C([1, 2])$ but $(C([1, 2]), A, x^*)$ does *not* have property SAIN (since if e is the unit function, then any $y\in A$ which satisfies $x^*(y)=x^*(e)=1$ must necessarily satisfy $\|y\|>1=\|e\|$).

EXAMPLE 3. Let

$$L = \{x \in C([0, 1]): x'(0) \text{ exists}, x'(0) = x(0)\}.$$

Then L is a *dense linear sublattice* in $C([0, 1])$ but $(C([0, 1]), L, \delta_0)$ does *not* have property SAIN (since if e is the unit function and y is any element of L satisfying $y(0)=e(0)=1$, then $y'(0)=y(0)=1$ and so $y(t)>1$ for some $t>0$ and hence $\|y\|>1=\|e\|$).

In the case when $X=L_p=L_p(T, \Sigma, \mu)$ $(1<p<\infty)$ and M is the subspace of L_p consisting of those functions which vanish off sets of finite measure, we can prove the following theorem. (Recall that the *representer* of a functional $x^*\in L_p^*$ is the function $y\in L_q$, $q=p/(p-1)$, such that $x^*(x)=\int_T xy\, d\mu$ for all $x\in L_p$.)

THEOREM 3. *Let* $1<p<\infty$, *let* $M\subset L_p$ *be as above, and let* $x_1^*, \cdots, x_n^*\in L_p^*$. *Then the following statements are equivalent.*

(1) $(L_p, M, \{x_1^*, \cdots, x_n^*\})$ *has property SAIN.*

(2) *Each* x_i^* *attains its norm on the unit ball in* M.

(3) *The representer of each* x_i^* *vanishes off a set of finite measure.*

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