ON THE SPACE OF HOMEOMORPHISMS OF E^3

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Introduction. Hamstrom [4] has shown that the space $\mathcal{H}(M)$ of homeomorphisms of a compact 3-manifold M with boundary is LC^n for $n=0, 1, 2, \cdots$, where $\mathcal{H}(M)$ has the compact-open topology. Kister [7] has shown that for a 3-manifold M with boundary, $\mathcal{H}(M)$ is LC^0 if $\mathcal{H}(M)$ is topologised by the metric ρ^* given by

$$\rho^*(f, g) = \sup_{x \in M} \rho(f(x), g(x))$$

where ρ is the natural metric for some locally finite triangulation of M. He has also shown [6] that $\mathcal{H}(E^n)$ is locally contractible in the topology induced in $\mathcal{H}(E^n)$ by the usual Euclidean metric.

However, if M is not compact, different metrics on M give rise to different topologies on $\mathcal{H}(M)$ in some of which $\mathcal{H}(M)$ is not LC^0. Fort [5] has shown that if P is the plane then $\mathcal{H}(P)$ is LC^0 if $\mathcal{H}(P)$ has the compact-open topology. In this note we extend his result to E^3.

Results.

THEOREM. *$\mathcal{H}(E^3)$ with the compact-open topology is LC^n for $n=0, 1, 2, \cdots$.*

PROOF. With the compact-open topology $\mathcal{H}(E^3)$ is a topological group and we need only prove the assertion that $\mathcal{H}(E^3)$ is LC^n at the identity i.

Let U be a neighbourhood of i. Then there exists an open set V of the form $\bigcap_{i=1}^{n} (A_i, V_i)$, where (A_i, V_i) is the set of all elements of $\mathcal{H}(E^3)$ which map the compact set A_i into the open set V_i, such that $i \in V \subset U$. There exists an $\epsilon > 0$ such that the ϵ-neighbourhood of A_i is contained in V_i, $i=1, 2, \cdots, n$.

There exist geometric balls B_1 and B_2 with $\bigcup_{i=1}^{n} V_i \subset B_1 \subset \text{Int } B_2$. By Theorem 5.1 of [4] there exists a $\delta > 0$ such that any mapping f of S^m into the space of homeomorphisms of B_1 into B_2 that move no point as much as δ, can be "extended" to a mapping F' of S^m into the space of homeomorphisms of B_2 onto itself which leave Bd B_2 pointwise fixed. This can be done so that each $F'(s)$ is an extension of $f(s)$ and $F'(s)$ moves no point as much as $\epsilon/2$. We can define a mapping $F: S^m \to \mathcal{H}(E^3)$ by $F(s)|B_2 = F'(s)$ and $F(s)|E^3 - B_2 = \text{id}|E^3 - B_2$.

Using Alexander's Theorem [1] as given in [2] we can define a homotopy H of F onto the identity map such that $H(s, t)$ moves no point as much as $\epsilon/2$. We can cover B_1 with a finite number of closed $\delta/4$ balls, $V'_1, V'_2, \cdots, V'_k$. Then the open set $W=\bigcap_{i=1}^{k} (V'_i, S(V'_i, \delta/4))$, where $S(V'_i, \delta/4)$ is the open $\delta/4$-neighbourhood of V'_i, is a neighbourhood of the identity in $\mathcal{H}(E^3)$. If f is a map of S^m into W then the "restriction" $f|B_1$ defined by $f|B_1(s)=f(s)|B_1$ is a mapping of S^m into the space of homeomorphisms of B_1 into B_2 which move no point as much as δ. Then, as above, there is an "extension" F of $f|B_1$ which maps S^m into $\mathcal{H}(E^3)$ and a homotopy H of F onto the identity such that

$$H(s, 0) = \text{identity}, \qquad H(s, 1) = F(s),$$

and $H(s, t)\in U$ for all $s\in S^m$ and $t\in I$.

Define $K(s, t)$ to be $(H(s, t))^{-1}f(s)$. Then $H(s, 0)$ is $f(s)$, and $H(s, 1)=\bar{f}(s)$ is the identity on B_1 for all $s\in S^m$. f is homotopic to $\bar{f}$ in U.

Define a homotopy G of $\bar{f}$ onto the identity by

$$G(s, t)(x) = 1/t\bar{f}(s)(tx), \qquad t \neq 0$$

and

$$G(s, 0)(x) = x.$$

Therefore f is homotopic to the identity and $\mathcal{H}(E^3)$ is LC^n for $n=0, 1, 2, \cdots$. In the same way, using conformal mapping theory (see [3] for details), it can be shown that $\mathcal{H}(P)$, with the compact-open topology, is locally contractible.

References

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