ON EXISTENCE AND RIGIDITY OF ISOMETRIC IMMERSIONS

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Introduction. In the study of the geometry of a hypersurface in euclidean space, the two fundamental results are the existence theorem and the rigidity theorem. The existence theorem states that a simply connected Riemannian manifold equipped with a second fundamental form for which the Gauss and Codazzi-Mainardi equations hold can be realized as a codimension one immersed submanifold of euclidean space. The rigidity theorem asserts (roughly) that any two such realizations differ by a rigid motion of the containing euclidean space. The effect of these two theorems is to reduce the study of immersed hypersurfaces in euclidean space to the study of Riemannian manifolds equipped with second fundamental forms satisfying the Gauss and Codazzi-Mainardi equations.

The purpose of this paper is to generalize these two results to isometric immersions of Riemannian manifolds in euclidean space with arbitrary codimension (always greater than zero). Our existence theorem is an analogue of the result of Hirsch [2] for smooth immersions. It states that a simply connected Riemannian manifold which has a k-plane bundle over it equipped with a bundle metric, a compatible connection, and a second fundamental form for which the Gauss and Codazzi-Mainardi equations hold can be isometrically immersed in euclidean space of codimension k. The rigidity theorem asserts that the normal bundle of an isometric immersion together with its induced bundle metric, connection, and second fundamental form essentially determine the immersion up to a rigid motion of the euclidean space.

It should be remarked here that the techniques used to prove the above results also apply to isometric immersions in spheres and hyperbolic spaces. Results of this type and detailed proofs of the results announced here will appear in [4].

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1. Statement of results. We begin by putting into a bundle setting a well-known necessary condition on the existence of an isometric immersion of a Riemannian manifold in euclidean space. In what follows, all objects (maps, manifolds, bundles, etc.) will be differentiable of class C^∞.

Let $\phi: M\rightarrow R^{n+k}$ be an isometric immersion of a Riemannian n-manifold into euclidean space, $\pi: E\rightarrow M$ it's normal bundle, and $T=TM$ the tangent bundle of M. As the normal bundle of an immersion, E possess additional structure which we now describe.

First of all, E has a *bundle metric* defined in the obvious way. Explicitly, the inner product of two normal vectors at a point is their inner product in the euclidean space.

Secondly, E has an *affine connection* D where, for $X\in TM_p$ and N a normal field on M, D_XN is obtained by taking the covariant derivative of N with respect to $d\phi X$ in euclidean space and projecting onto the normal plane $E_p=\pi^{-1}(p)$ through $\phi(p)$. It is easy to show that this connection is compatible with the metric in E defined above; that is

$$X(N, N') = (D_XN, N') + (N, D_XN') \tag{1.1}$$

where N, N' are normal fields on M and $X\in TM_p$. (Since no confusion seems likely, we use the notation (,) both for the inner product in E and for the inner product on TM defined by the Riemannian structure.)

Finally, E has a *second fundamental form* A which is a section in the bundle Hom $(T\otimes E, T)$. The definition of A_XN proceeds exactly as the above definition of D_XN except that we project onto the tangent plane thru $\phi(p)$ instead of the normal plane. It is routine to prove that

$$(A_XN, Y) = (X, A_YN) \tag{1.2}$$

for any tangent vectors X and Y on M and normal vector N. We define the *second fundamental tensor associated with* A to be the section B in Hom $(T\otimes T, E)$ defined by

$$(B(X, Y), N) = (A_XN, Y) \tag{1.3}$$

where X, Y are tangent vectors on M and N is a normal vector. Clearly B is symmetric.

As usual, we denote the *Riemannian curvature* tensor by R and define $\overline{R}$, the *curvature of E relative to D*, by the equation

$$\overline{R}(X, Y)N = D_XD_YN - D_YD_XN - D_{[X,Y]}N.$$

In this context, the *Gauss equations* are

$$R(X, Y)Z = A_X B(Y, Z) - A_Y B(X, Z)$$

and

$$\bar{R}(X, Y)N = B(A_X N, Y) - B(X, A_Y N)$$

and the *Codazzi-Mainardi equations* are

$$\nabla_X A_Y N - \nabla_Y A_X N - A_{[X,Y]} N = A_Y D_X N - A_X D_Y N.$$

In the above, ∇ is the Levi-Civita connection on M, X, Y, and Z are tangent vector fields on M and N is a section in E (a normal field).

It is well known (see, for example, Hicks [1, p. 76]) that the Gauss and Codazzi-Mainardi equations are satisfied in the above situation. Thus, the existance of a k-plane bundle over a Riemannian manifold with the additional structure described above for which the Gauss and Codazzi-Mainardi equation hold is a necessary condition for the existence of an isometric immersion in R^{n+k}. Our existence theorem below asserts that, if M is simply connected, this condition is also sufficient.

We call a k-plane bundle over a manifold a *Riemannian k-plane bundle* if it is equipped with a bundle metric and compatible connection. If E is any k-plane bundle over a Riemannian manifold M, a *second fundamental form* in E is a section A in $\mathrm{Hom}(T\otimes E, T)$ satisfying (1.2). If E is a Riemannian vector bundle with a second fundamental form A, we define the *associated second fundamental tensor* B as in (1.3).

We can now state our main results.

EXISTENCE THEOREM. *Let M be a simply connected Riemannian n-manifold with a Riemannian k-plane bundle E over M equipped with a second fundamental form A and associated second fundamental tensor B. Then, if the Gauss and Codazzi-Mainardi equations are satisfied, M can be isometrically immersed in R^{n+k} with normal bundle E.*

RIGIDITY THEOREM. *Let ϕ, ϕ': $M \to R^{n+k}$ be isometric immersions of a Riemannian n-manifold with normal bundles E, E' equipped as above with bundle metrics, connections, and second fundamental forms. Suppose there is an isometry f: $M \to M$ that can be covered by a bundle map $\bar{f}$: $E \to E'$ which preserves the bundle metrics, the connections, and the second fundamental forms. Then there is a rigid motion F of R^{n+k} such that $F\circ\phi = \phi'\circ f$.*

REMARKS. a. *Note that we do not require in the existence theorem that the Whitney sum $T\otimes E$ be trivial. This in fact follows from the simply connectivity of M and the fact that (as we show below) a neighborhood* *of the zero section in E admits a flat Riemannian metric.* (See Milnor [3].)

b. *In the rigidity theorem, it is not necessary that M be simply connected.*

2. **The proofs of the main theorem.** In this section, we sketch the proofs of the existence and rigidity theorems stated in the previous section. The basic idea is to reduce the general case to the codimension zero problem for flat manifolds. We begin by considering the codimension one case for a compact simply connected manifold since the idea of the proof is most clearly illustrated in this case. We conclude the section by indicating the adjustment necessary to prove the results in general.

Suppose $\phi: M \to R^{n+1}$ is an isometric immersion of a compact simply connected Riemannian n-manifold. Choose a unit normal field N on M and $\epsilon > 0$ so that the map

$$\tilde{\phi}: U = MX(-\epsilon, \epsilon) \to R^{n+1}$$

defined by $\tilde{\phi}(x, t) = \phi(x) + tN$ is an immersion. Since $\tilde{\phi}$ immerses U as an open subset of R^{n+1}, the Riemannian metric induced on U is flat. An easy computation shows that the metric is given as follows.

LEMMA 2.1. *Identifying $TU_{(x,t)}$ with $TM_x \times R$, the inner product $\langle\ ,\ \rangle$ on $TU_{(x,t)}$ induced by $\tilde{\phi}$ is the direct sum of the obvious metric on R (obtained by identifying R with the one dimensional subspace of R^{n+1} generated by N_x) with the metric*

$$\langle X, Y\rangle = (X + A_X tN,\ Y + A_Y tN)$$

where $X, Y \in TM_x$, $(\ ,\ )$ is the inner product on TM_x, and A the second fundamental form of the immersion ϕ.

To prove the existence theorem in this case, suppose E is a line bundle over M equipped as in the existence theorem. Choose a unit section N in E, define a metric on the manifold E as in Lemma 1.2, and let U be the tubular neighborhood of the zero section in E on which the metric is nonsingular. This metric obviously induces the original metric on M.

The following two lemmas complete the proof of the existence theorem.

LEMMA 2.2. *If the Gauss and Codazzi-Mainardi equations are satisfied, the metric defined above on U is flat.*

LEMMA 2.3. *Suppose V and W are flat Riemannian m-manifolds, V* *simply connected, and W complete. Then there is an isometric immersion of V into W.*

The first of these lemmas follows from a straightforward computation and is given (in full generality) in [4]. The second lemma is essentially well known.

The proof of the rigidity theorem is easier. Suppose $\phi, \phi': M \to R^{n+1}$ are isometric immersions and $f: M \to M$ isometry preserving second fundamental forms. By Lemma 2.1, we can extend f to an isometry $\bar{f}: U \to U$ (where ϕ and ϕ' extend to immersions $\bar{\phi}, \bar{\phi}': U \to R^{n+1}$). Let $V \subset U$ be a coordinate ball on which both $\bar{\phi}$ and $\bar{\phi}' \circ \bar{f}$ are embeddings and F a rigid motion of R^{n+1} such that $F \circ \bar{\phi} | V = \bar{\phi}' \circ \bar{f} | V$. It then follows easily that $F \circ \bar{\phi} = \bar{\phi}' \circ \bar{f}$ so that $F \circ \phi = \phi' \circ f$.

The only significant adjustment needed to prove the theorems in arbitrary codimension is in Lemma 2.1. Suppose $\phi: M \to R^{n+k}$ is an isometric immersion with normal bundle $\pi: E \to M$ and U a tubular neighborhood of the zero section in E for which ϕ extends to an immersion $\bar{\phi}: U \to R^{n+k}$. Let $TE \simeq H \oplus V$ be the decomposition into horizontal and vertical subbundles defined by the connection D where $H \simeq \pi^* TM$ and $V \simeq \pi^* E$. The appropriate generalization of Lemma 2.1 is the following.

LEMMA 2.4. *The metric on $TU \simeq H \oplus V$ induced by the immersion $\bar{\phi}$ is the direct sum of the metric on V induced by the equivalence $V \simeq \pi^* E$ from the given metric on E with the metric $\langle\ ,\ \rangle$ on H given by*

$$\langle X', Y' \rangle = (X + A_X Z, Y + A_Y Z)$$

where $X', Y' \in TU_Z$, $X = d\pi X'$, $Y = d\pi Y'$, A is the second fundamental form of the immersion, and $(\ ,\ )$ is the metric on TM.

The proof of this theorem is given in [4]. The remainder of the proofs of the existence and rigidity theorems proceed essentially as in the codimension one case.

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