A COUNTABILITY CONDITION FOR PRIMARY GROUPS PRESENTED BY RELATIONS OF LENGTH TWO[1]

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A subgroup A of the p-primary group G is *nice* if $p^\alpha(G/A) = \{p^\alpha G, A\}/A$ for all ordinals α. We consider the following countability condition: there exists a collection $\mathcal{C}$ of nice subgroups of G such that

(0) $0 \in \mathcal{C}$.

(1) $\mathcal{C}$ is closed with respect to group-theoretic union in G.

(2) If $A \in \mathcal{C}$ and if H is a subgroup of G such that $\{A, H\}/A$ is countable, there exists $B \in \mathcal{C}$ such that $B \supseteq \{A, H\}$ and such that B/A is countable.

The author [1] has referred to this condition as the *third axiom of countability* and has demonstrated that this is the countability condition—not the *first axiom* (countability) nor the *second axiom* (decomposition into a direct sum of countable groups)—which is truly relevant for the proof of Ulm's theorem.

In this note, we outline a short proof of

THEOREM. *Suppose that G is a p-primary group presented by an arbitrary number of generators x_i ($i \in I$) and relations $R_m (m \in M)$. If each relation R_m involves at most two generators, then G satisfies the third axiom of countability.*

PROOF. There is, of course, no loss of generality in assuming that the index sets I and M both contain an element denoted by 0 and that the relation R_0 is: $x_0 = 0$. By adding repeatedly, if necessary, new generators y_i subject to defining relations of the form $px_i = y_i$, we may assume that for each $i \neq 0$ in I that there exists a relation R_m of the form $px_i = x_j$. Since each element of G has order equal to a power of p, we may in fact assume that given any generator $x_1 \neq x_0$ having order p^n in G that there exists a finite chain $x_1, x_2, x_3, \cdots, x_{n+1} = x_0$ of generators such that $px_i = x_{i+1}$ is one of the given relations. Furthermore, by deleting certain redundancies in both generators and relations we may assume, in the end, that each relation R_m, $m \neq 0$, is precisely of the form $px_i = x_j$ where $i \neq j$. For a quick verification of this, note that if

$$(1) \qquad rx_i = sx_j \quad \text{where } i \neq j \quad \text{and} \quad (r, p) = 1$$

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then $x_i = tx_j$ for some integer t and that the generator x_i is redundant for the presentation of G—with relations involving at most two generators. Having removed the redundancy (1) from the presentation of G, we observe that each relation R_m, $m \neq 0$, is implied by the given relations of the form $px_i = x_j$, $i \neq j$, and $x_0 = 0$. Consider the relation

$$R_m: rx_i = sx_j,$$

where r and s are integers. This relation is equivalent to, in the presence of the relations $px_i = x_j$ and $x_0 = 0$, a relation of the form

$$R'_m: qx_\lambda = tx_\mu \quad \text{where } (q, p) = 1;$$

even if $r = 0$, $rx_i = 1x_0$. The denial of (1) implies that $\lambda = \mu$, so the relation R_n gives, beyond the information already given by the relations $px_i = x_j$ and $x_0 = 0$, information only about the order of some generator x_λ. However, the order of each generator x_i is already determined by the relations $px_i = x_j$ and $x_0 = 0$.

We have shown that G can be presented by generators x_i $(i \in I)$ and relations of the form $px_i = x_j$, $i \neq j$, and $x_0 = 0$ such that

(i) if $i \neq 0$ is in I, there exists $j \in I$ such that $px_i = x_j$ is a relation;

(ii) there is no redundancy of the form (1).

The proof of the theorem is clearly finished by the following lemma—G has plenty of nice subgroups.

LEMMA. *Suppose that the primary group G is presented by generators $x_i(i \in I)$ and relations as described above. Let $J \subseteq I$ and let H be the subgroup of G generated by the generators x_j, $j \in J$. Then H is a nice subgroup of G.*

PROOF. Define inductively for each ordinal α a subset I_α of I in the following way: $I_0 = I$,

$$I_{\alpha+1} = [i \in I: px_j = x_i \quad \text{for some } j \in I_\alpha]$$

and

$$I_\beta = \bigcap_{\alpha<\beta} I_\alpha \quad \text{if } \beta \text{ is a limit ordinal.}$$

It is easy to prove inductively that $p^\alpha G = \{x_i\}_{i \in I_\alpha}$. In order to show that H is a nice subgroup of G, we need to prove $p^\alpha(G/H) = \{p^\alpha G, H\}/H$ for all α. This, too, is an inductive argument and the nonlimit case is trivial. Suppose that β is a limit and assume that $p^\alpha(G/H) = \{p^\alpha G, H\}/H$ for all $\alpha < \beta$. Let $x + H \in p^\beta(G/H)$. We may assume that $i \in J$ if $px_i = x_j$ where $j \in J$. Write

$$x = \sum_{j \in J} t_j x_j + \sum_{k \in K} t_k x_k, \quad \text{where } (t_i, p) = 1$$

and K is disjoint from J. By the induction hypothesis, for each $\alpha<\beta$, there exists $h^\alpha=\sum_{j\in J} s_j^\alpha x_j$ in H such that $x+h^\alpha\in p^\alpha G=\{x_i\}_{i\in I_\alpha}$. Thus

$$\sum_{j\in J}(t_j+s_j^\alpha)x_j+\sum_{k\in K}t_kx_k=\sum_{i\in I_\alpha}r_ix_i.$$

Since $(t_k, p)=1$ for each $k\in K$, it follows immediately from the limited substitution, $px_i=x_j$, one can make on generators that $k\in I_\alpha$ for each $k\in K$. Hence $k\in I_\beta=\bigcap_{\alpha<\beta} I_\alpha$ for each $k\in K$. Defining $x'=\sum_{k\in K} t_kx_k$, we have that $x'\in p^\beta G$ and $x+H=x'+H\in\{p^\beta G, H\}/H$. This verifies that H is nice in G.

In connection with our result, the reader's attention is called to [2]; the connection should be obvious.

REFERENCES

1. Paul Hill, *Ulm's theorem for totally projective groups*, Notices Amer. Math. Soc. **14** (1967), 940.

2. Peter Crawley and Alfred Hales, *The structure of torsion abelian groups given by presentations*, Bull. Amer. Math. Soc. **74** (1968), 954–956.

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