RULED SURFACES AND THE ALBANESE MAPPING

BY ARTHUR MATTUCK[1]

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1. Much of the classical theory of algebraic curves is summarized by saying there is a map $C(n) \to J$ from the n-fold symmetric product of the curve C into an abelian variety J, the Jacobian, and the fibers are projective spaces (representing the linear systems of degree n). For algebraic surfaces there is an analogous map $V(n) \to A$ from the n-fold symmetric product of the surface V to its Albanese variety. The fibers are irreducible and regular if n is large, but it has been a long open question whether they are rational, or ever can be.

THEOREM. *Let V be a complete nonsingular surface in characteristic zero, and let q denote the dimension of its Albanese variety A. If for some $n > q$ the general fiber of the morphism $V(n) \to A$ is a rational variety, then V is a ruled surface.*

By the "general" fiber we mean as usual that there is an open set in A over which all fibers have the indicated property. If V is ruled, i.e., birationally equivalent to the product $P^1 \times C$ of a projective line and a curve C, then the general fiber is rational for all n: for this converse to the theorem, one needs only the quoted result for curves plus the remark that then the Albanese variety of V is just the Jacobian of C. A proof of the theorem when $q = 0$ was the subject of an earlier paper [2], some of whose ideas recur here. There is also overlap with a recent (independent) proof by Mumford [3] that the rational equivalence ring is not of finite type; both proofs use the idea of bounding the dimension of the zero-locus of a 2-form.

2. **A generic smoothness lemma.** We need the

LEMMA. *Let $f: X \to Y$ be a dominating morphism of varieties in characteristic zero, with X nonsingular and projective. Then f has maximal rank along the general fiber F_y, so F_y is nonsingular.*

PROOF. The lemma is local on Y; by Noether normalization we may reduce to the case where Y is affine r-space, with coordinate functions $x_1, \cdots, x_r$. As a_1 varies over the (algebraically closed) ground field, the zeros of $x_1 - a_1$ on X give a linear system of divisors on X; by Bertini's theorem, a general member—say X_1—is a disjoint union of

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nonsingular varieties; each has multiplicity one and no embedded components. In the same way the zeros of $x_2 - a_2$ on X_1 give a linear system on X_1 whose general member is nonsingular, etc. Continuing, we see that there exist fibers $f^{-1}(a)$ along which f is of maximal rank. This then follows for the general fiber since f being of maximal rank is an open condition on Y: it is clearly open on X and f is proper.

For the proof of our theorem, we may replace the singular variety $V(n)$ by a canonical desingularization, the Hilbert scheme $H(n)$. This is [1] a nonsingular $2n$-dimensional projective variety with a birational morphism $h_n: H(n) \to V(n)$. This gives a corresponding morphism $H(n) \to A$, whose general fiber is pure $(2n-q)$-dimensional by dimension theory, and it is easily seen that h_n is an isomorphism on an open subset of each component of the general fiber. Thus our hypotheses together with the lemma imply that

(1) *For some $n > q$, the general fiber of $H(n) \to A$ is a nonsingular rational variety.*

3. **Differential forms of weight r.** Let X be an n-dimensional variety, with function field $K = k(X)$, and let E^p be the K-space of p-forms on X; then the $(m_1, \cdots, m_r)$*-forms* are the elements of $E^{m_1} \otimes \cdots \otimes E^{m_r}$; if all $m_i = m$, they are called the *m-forms of weight r*. These forms are *holomorphic at $p \in X$* if the coefficients are holomorphic when the form is written in terms of $dx_1, \cdots, dx_n$, where the x_i are local parameters at p. Thus if X is also complete, the number of independent global holomorphic $(m_1, \cdots, m_r)$-forms is given by $h^{m_1, \cdots, m_r} = \dim H^0(X, \Omega^{m_1} \otimes \cdots \otimes \Omega^{m_r})$, where Ω^k is the sheaf of holomorphic k-forms. If $m_i = n$ for all i, $h^{n, \cdots, n}$ is traditionally written $P_r(X)$, and called the *rth plurigenus* of X. These are all birational invariants for X complete nonsingular, for

(2) *If $f: X \to Y$ is a dominating, separable, rational map of complete nonsingular varieties, then $h^{m_1, \cdots, m_r}(X) \geq h^{m_1, \cdots, m_r}(Y)$.*

The reasoning is classical. If α is a holomorphic form on Y, then $f^*\alpha$ is a form of the same type on X which is nonzero (separability); holomorphic outside a locus of codimension ≥ 2 (the fundamental locus), therefore holomorphic everywhere (nonsingularity of X).

PROPOSITION. *If X is a nonsingular rational (or unirational) variety, then $h^{m_1, \cdots, m_r}(X) = 0$ for all $(m_1, \cdots, m_r) \neq (0, \cdots, 0)$.*

PROOF. It suffices to prove this when X is projective n-space, by (2). Let $x_0, \cdots, x_n$ be projective coordinates, and $g: A - (0) \to X$ the usual map of affine $(n+1)$-space minus the origin onto projective space. If α is a holomorphic form on X, then $g^*\alpha$ is holomorphic on $A-(0)$, therefore on A since cod $(0)\geqq 2$. Written in terms of x_i, its coefficients are thus polynomials; since it is invariant under the automorphisms of A defined by $x_i \to cx_i$, we get all $m_i=0$.

4. **Proof of the theorem.** By a well-known result (see e.g. [4]), if V is a complete nonsingular surface in characteristic zero, then V is ruled if and only if $P_r(V)=0$ for all $r>0$. So we prove:

(3) *If for some r, V carries a nonzero holomorphic 2-form ϕ of weight r, then the general fiber of $H(n)\to A$ is not rational.*

Let $V[n]$ be the n-fold product; given such a ϕ, then

$$\Phi = \phi_1 + \cdots + \phi_n, \qquad \phi_i = \mathrm{pr}_i^*\phi \tag{4}$$

is a holomorphic 2-form of weight r on $V[n]$; since it is invariant under the symmetric group S_n, it is the lifting of a form on $V(n)$, and this in turn may be carried over to $H(n)$. We use the same letter Φ for any of these forms. If we grant that Φ is holomorphic on $H(n)$—this will be proved later—then the restriction Φ_F of Φ to a (nonsingular) general fiber F of $H(n)\to A$ gives a holomorphic 2-form of weight r on F. If F were rational, then $\Phi_F=0$ by the proposition. But if we pull things back to $V[n]$, this contradicts

(5) *If $n>q$, the restriction of Φ to the general fiber of $V[n]\to A$ is not zero.*

Proof of (5). Let $p=(p_1, \cdots, p_n)$ be a general point of $V[n]$, F the fiber through it, $T_{p,F}$ the tangent space to F at p. We say

(6) *$\sigma_i\colon T_{p,F}\to T_{p_i,V}$ is onto for all i ($\sigma_i=d(pr_i|F)$).*

Namely, let S_i be the closure of the set of $q\in V[n]$ which are either singular points of the fiber F_q through them or else where σ_i is not onto, i.e., has rank $\leqq 1$. Since dim $T_{p,F}=2n-q>n$, this space cannot be mapped to a 1-dimensional space by each of the n maps σ_i. Say σ_1 has rank 2; introducing coordinates, we see that rank $\sigma_1=2$ in a neighborhood of p. Thus $p\notin S_1$, so S_1 is a proper closed set. It follows by symmetry that S_i is a proper closed set, and therefore $p\notin S_i$ for any i, which is the assertion (6).

From (6) it follows that for each i, we can choose vectors t_i, t_i' in $T_{p,F}$ whose images under σ_i are independent. Taking general linear combinations of the t_i and of the t_i', we conclude

(7) *There are vectors t, t' in $T_{p,F}$ such that $\sigma_i(t)$ and $\sigma_i(t')$ are independent for all i.*

We now prove (5). Choose x and y to be local parameters at each point p_i; thus $\phi = g(dxdy)|^{\otimes r}$, where $g(p_i) = a_i \neq 0$ since p_i is a general point of V. By (7), $\langle dxdy, (\sigma_i(t), \sigma_i(t'))\rangle = b_i \neq 0$. On the space $T_{p,V[n]}$, by (4) the form $\Phi = \sum a_i(dx_i dy_i)|^{\otimes r}$. If Φ were 0 when restricted to the subspace $T_{p,F}$, then for $e, e' \in T_{p,F}$,

$$\langle \Phi, (e, e', t, t', \cdots, t, t')\rangle = \sum a_i \langle dx_i dy_i, (e, e')\rangle b_i^{r-1} = 0.$$

Our hypothesis is that dim $T_{p,F} > n$. If we put in $n+1$ linearly independent vectors for e', we get from the above $n+1$ independent linear equations in $2n$ variables (the coefficients of e), having at least $n+1$ independent solution vectors e, a contradiction.

We still must show Φ is holomorphic on $H(n)$. Let X be the normalization of $H(n)$ in the function field of $V[n]$. Then the symmetric group S_n acts as automorphisms of X and $H(n)$ is the quotient X/S_n. Since Φ is holomorphic on $V[n]$, when viewed as a differential Φ' on the normal and birationally equivalent variety X, it will have no poles. Therefore on $H(n)$, its trace $\mathrm{tr}_{X|H(n)}\Phi'$ will also have no poles; but $(1/n!)$ tr $\Phi' = \Phi$.

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Massachusetts Institute of Technology, Cambridge, Massachusetts 02139