TRIANGULATION OF MANIFOLDS. II

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Main theorems. We will say that a manifold M satisfies condition S, if $\pi_1(M\times T^k)$ and $\pi_1(\partial M\times T^k)$ satisfy the conditions necessary for the splitting theorem to hold [6], [9].

THEOREM 4. *Every closed topological manifold M, dim $M\geqq 5$, $H^4(M; Z_2)=0$, and satisfying condition S, admits a PL manifold structure.*[2]

PROOF. By Theorem 3 and addendum to Theorem 2, the tangent bundle of M^n lifts to a PL_n-bundle. By the splitting theorem [6], [9], there is a PL-manifold Q of the same tangential homotopy type as M. As in [5], proof of (c), we may immerse $M_0=M$-point in Q, to give M_0 a PL manifold structure. By Lees' Lemma [5], M admits a PL manifold structure.

REMARKS. 1. If we are given a lift of $\tau(M^n)$ to a PL_n-bundle, we may drop the condition $H^4(M; Z_2)=0$.

2. If we are given a bundle map of $\tau(M_0)$ into $\tau(Q)$, Q^n a PL manifold, we may drop condition S as well.

THEOREM 5. *Let W^n, $n\geqq 5$, be a topological h-cobordism between PL manifolds. If $H^3(W; Z_2)=0$, then W admits a PL manifold structure with the given structures on the boundary.*

PROOF. Say $\partial W=M_1\cup M_2$. Then we may define inclusions ι_1: $M_1\times I\rightarrow W$, ι_2: $M_2\times I\rightarrow W$ using collar neighborhoods. (Take $\iota_1|M_1\times 0=$identity and $\iota_2|M_2\times 1=$identity.) Also we have retractions r_1: $W\rightarrow M_1\times I$, r_2: $W\rightarrow M_2\times I$, where for example we may take $r_2|M_2$: $M_2\rightarrow M_2\times 1$ by the identity, $r_2|M_1$: $M_1\rightarrow M_2\times 0$ by a homotopy equivalence, and $r_2\iota_2=$identity. Now these maps are covered by topological bundle maps; ι_i^*: $\tau_1\oplus 1\rightarrow\tau=\tau(W)$, ι_2^*: $\tau_2\oplus 1\rightarrow\tau$, and r_2^*: $\tau\rightarrow\tau_2\oplus 1$ so that $r_2^*\iota_2^*=$identity (since $M_2\times I$ is a deformation retract of W). Then $r_2^*\iota_1^*$: $\tau_1\oplus 1\rightarrow\tau_2\oplus 1$ is a topological bundle map.

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[2] As first shown by Kirby and Siebenmann (by other methods), condition S may be eliminated. We can do this by applying Theorem 7 below to the normal disk bundle of a compact manifold M (condition 3 is unnecessary since the tangent bundle is trivial) to obtain their result that M is the homotopy type of a finite complex. The splitting theorem then holds with no condition on the fundamental group [9].

Now assume $r_2^* \iota_1^*$ is isotopic to a PL bundle map. Enlarge W to the open $W' = M_1 \times (-1, 0] \cup W \cup M_2 \times [1, 2)$. Then using Lees' Theorem we may immerse a neighborhood of W in $M_2 \times R$ by the bundle map $r_2^*: \tau \to \tau_2 \oplus 1$. Since $r_2^* \iota_2^* =$ identity and $r_2^* \iota_1^*$ is isotopic to a PL bundle map, we may assume the immersion ϕ satisfies $\phi_2: M_2 \times [0, 1] \to M_2 \times R$ is the inclusion, and $\phi\iota_1: M_1 \times [0, 1] \to M_2 \times R$ is a PL immersion. Thus the immersion defines a PL structure on W, which agrees with the given structures on the boundary.

To show $r_2^* \iota_2^*$ is isotopic to a PL bundle map under the hypothesis that $H^3(W, Z_2) = H^3(M_1; Z_2) = 0$; note that topological (PL) bundle maps over $r_2 \iota_2$ are given by cross-sections of an associated bundle E^{Top} (E^{PL}) over M_1 with fibre Top_n (PL_n). The map $\mathrm{PL}_n \to \mathrm{Top}_n$ induces a map $\rho: E^{\mathrm{PL}} \to E^{\mathrm{Top}}$. The fibre of ρ is homotopy equivalent to $\Omega(\mathrm{Top}_n/\mathrm{PL}_n)$, which has at most one nontrivial homotopy class in dimensions $\leqq n-2$. This is in dimension 2, and at most of order 2. Thus the only obstruction to lifting a cross-section of E^{Top} to one of E^{PL} lies in $H^3(M_1; Z_2) = H^3(W; Z_2)$.

THEOREM 6. *Let M^n be a compact topological manifold with boundary N^{n-1}, with fundamental groups satisfying conditions S.*

(a) *If $H^4(M, Z_2) = H^3(N, Z_2) = 0$, and $n \geqq 6$, M admits a PL manifold structure.*

(b) *If N already has a PL structure, $H^4(M, Z_2) = H^3(N, Z_2) = 0$ and $n \geqq 5$, then M admits a PL manifold structure agreeing with the given one on the boundary.*

PROOF. (a) If $H^4(M; Z_2) = 0$, then $\tau(M)$ lifts to a PL_n bundle. This induces a lift of $\tau(N) \oplus 1$ to a PL_n bundle; which lifts in turn to a PL_{n-1} bundle α over N, since $\pi_i(\mathrm{PL}_n, \mathrm{PL}_{n-1}) = 0$, for $i \leqq n-2$. The lift is unique, except on the top cell of N. Since $\pi_{n-1}(\mathrm{PL}_n, \mathrm{PL}_{n-1}) \simeq \pi_{n-1}(\mathrm{Top}_n, \mathrm{Top}_{n-1})$ (see proof of Theorem 3 of I), we may choose α to be a lift of $\tau(N)$. As in the proof of Theorem 1, N may be triangulated. This reduces case (a) to case (b).

(b) Take the double of M. It admits a PL structure by Theorem 4. Hence $W = M - N$ has a PL structure. By (9), W is collared. $W = \overline{W} \cup | \partial \overline{W} \times [0, 1)$. Hence $M = \overline{W} \cup V$ where V is an h-cobordism between $\partial \overline{W}$ and ∂M. By Theorem 2, V admits a PL structure which extends the one on $\partial V = \partial \overline{W} \cup N$. Q.E.D.

Instead of using the splitting theorem to construct a PL manifold of the same tangential homotopy type, one may use the surgery techniques of Browder, Novikov, and Wall [12]. As an example we get

THEOREM 7. *Let M^n be a compact connected manifold with boundary N, such that*

1. *each component of N except one, N_0, has a PL structure.*
2. *$\pi_1(N_0) \to \pi_1(M)$ is an isomorphism.*
3. *$H^3(N; Z_2) = H^4(M; Z_2) = 0$.*

Then M admits a PL structure which extends the given ones on the boundary components.

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