takes, we also take comfort in the following thought attributed to a Chinese philosopher: 'If you write a letter to a friend, be certain to include at least one mistake. In this way you will make him feel superior'." The author does include a few minor errors—just enough to make the reader happy—but in no way do they impair the usefulness of his book.

The book is designed for a one-semester graduate course and seems, aside from the above reservation about choice of material, to be well suited to its purpose. Parts of it have also been used successfully in courses for undergraduates who have some understanding of the fundamentals of linear algebra and of the rudiments of Euclidean topology.

The book has been translated into German by E. Heil and published in 1968 by the Hochschultaschenbücher-Verlag, Bibliographisches Institut, Mannheim.

VICTOR KLEE

Nonlinear two point boundary value problems, by Paul Bailey, Lawrence Shampine, and Paul Waltman. Math. in Science and Engineering, vol. 44, Academic Press, New York, 1968. xiii+171 pp. $9.50.

In the preface, the authors state "the purpose of this book is to describe as simply as possible a number of the ideas and methods which seem to be particularly helpful in the study of nonlinear boundary value problems for differential equations of second order." The authors are primarily interested in establishing existence and uniqueness theorems for solutions of two point boundary value problems on as large an interval as possible for second order differential equations of the form

$$y'' + f(t, y, y') = 0. \tag{1.1}$$

The only two point problems considered in this monograph are the *first boundary value problem* on $[a, b]$ where the value of the unknown function $y(t)$ is prescribed at each end of the interval

$$y(a) = A, \qquad y(b) = B, \tag{1.2}$$

and the *second boundary value problems* where the value of $y(t)$ is prescribed at one end and the slope is prescribed at the other, either

$$y(a) = A, \qquad y'(b) = m \tag{1.3}$$

or

$$y'(a) = m, \qquad y(b) = B. \tag{1.4}$$

Chapter 1 serves as an introduction to the monograph. Here the kinds of boundary value problems to be treated are described and the conditions to be imposed on $f(t, y, y')$ are given. It is always assumed that the function $f(t, y, y')$ is continuous in (t, y, y') at least in the interior of its domain. For most theorems, a generalized Lipschitz condition of the form

$$G_1(y - x, y' - x') \leqq f(t, y, y') - f(t, x, x') \leqq G_2(y - x, y' - x') \tag{1.5}$$

where

$$\begin{aligned} G_1(y, y') &= K_1 y + L_1 y' && y \geqq 0, \quad y' \geqq 0 \\ &= K_1 y + L_2 y' && y \geqq 0, \quad y' \leqq 0 \\ &= K_2 y + L_2 y' && y \leqq 0, \quad y' \leqq 0 \\ &= K_2 y + L_1 y' && y \leqq 0, \quad y' \geqq 0 \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} G_2(y, y') &= K_2 y + L_2 y' && y \geqq 0, \quad y' \geqq 0 \\ &= K_2 y + L_1 y' && y \geqq 0, \quad y' \leqq 0 \\ &= K_1 y + L_1 y' && y \leqq 0, \quad y' \leqq 0 \\ &= K_1 y + L_2 y' && y \leqq 0, \quad y' \geqq 0 \end{aligned} \tag{1.7}$$

is assumed. Assuming this generalized Lipschitz condition permits the authors to establish existence and uniqueness theorems on as large an interval as possible. If one only assumes a uniform Lipschitz condition, this is not possible. In this introduction, a section is devoted to standard results (stated without proof) for initial value problems—existence, uniqueness, extendability, and continuous dependence.

Chapter 2 is a brief chapter devoted to two elementary observations relating the existence and uniqueness intervals for the first boundary value problem to the corresponding intervals for the second.

The next five chapters (Chapters 3 thru 7) are devoted to developing various techniques for the systematic study of two point boundary value problems. In Chapter 3, Picard's iteration scheme and refinements of it are used to establish existence and uniqueness results using the point of view of contracting maps in a Banach space. A nice feature of this monograph is that (except for Chapter 2) each chapter is concluded by giving several examples illustrating the more important theorems of the chapter followed by a final section which comments on the results of the chapter and gives some references.

A second way to approach solutions of nonlinear boundary value problems is by means of nonlinear initial value problems. This approach is developed in Chapters 4, 6 and 7, culminating in Theorem 6.2, p. 96. This theorem states that if $f(t, y, y')$ is continuous and satisfies (1.5), then boundary value problem (1.1), (1.2) has a unique solution whenever $0<b-a<\alpha(L_2, K_2)+\beta(L_1, K_2)$. This result is best possible. α and β are defined as follows:

$$\begin{aligned} \alpha(L, K) &\equiv \frac{2}{(4K-L^2)^{1/2}} \cos^{-1} \frac{L}{2\sqrt{K}}, && 4K - L^2 > 0 \\ &\equiv \frac{2}{(L^2-4K)^{1/2}} \cosh^{-1} \frac{L}{2\sqrt{K}}, && 4K - L^2 < 0,\ L > 0,\ K > 0 \\ &\equiv \frac{2}{L}, && 4K - L^2 = 0,\ L > 0 \\ &\equiv +\infty, && \text{otherwise} \end{aligned} \tag{1.8}$$

$$\begin{aligned} \beta(L, K) &\equiv \frac{2}{(4K-L^2)^{1/2}} \cos^{-1} \frac{-L}{2\sqrt{K}}, && 4K - L^2 > 0 \\ &\equiv \frac{2}{(L^2-4K)^{1/2}} \cosh^{-1} \frac{-L}{2\sqrt{K}}, && 4K - L^2 < 0,\ L < 0,\ K > 0 \\ &\equiv -\frac{2}{L}, && 4K - L^2 = 0,\ L < 0 \\ &\equiv +\infty, && \text{otherwise.} \end{aligned} \tag{1.9}$$

In Chapter 7, modifications and extensions of the results of Chapter 6 are given. For example, the authors point out that

(a) the Lipschitz condition need not hold for all y;

(b) existence can hold on intervals $[a, b]$ which are too large to guarantee uniqueness, and

(c) K_i, L_i can depend on t in (1.5), (1.6), and (1.7).

Chapter 5 is devoted to comparison theorems for nonlinear equations. The best theorems proven in this chapter require that $h(t, y, y')$ be continuous on $[a, b]\times R\times R$, all initial value problems for $y''+h(t, y, y')+\epsilon=0$, ϵ sufficiently small, have unique solutions which exist throughout $[a, b]$, and solutions of the first boundary value problems for $y''+h(t, y, y')+\epsilon=0$, ϵ sufficiently small, are unique if they exist. Jackson and Schrader (Proc. Amer. Math. Soc. **17** (1966), 1023–1027) and Schrader (*On second order differential inequalities*, Proc. Amer. Math. Soc. (to appear)) have proven considerably sharper results, using elementary techniques.

The final two chapters consider numerical solutions by initial value methods and boundary value methods, respectively.

The monograph is written in a very clean style and is remarkably free of errors of any kind. The authors have achieved their goal of showing "how questions of existence and uniqueness of solutions of boundary value problems can be studied by methods that are entirely elementary yet rigorous." Although the authors did not attempt to cover the subject in depth, a third technique—subfunctions and differential inequalities—should be mentioned to those interested in problems in this area. A good recommended supplement to this monograph is a recent survey paper by Lloyd Jackson, *Subfunctions and second order ordinary differential inequalities*, Advances in Math. **2** (1968), 307–363.

J. W. BEBERNES