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ON SPHERE-BUNDLES. I

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Let E be an $(n-1)$-sphere bundle over a base space B, with the orthogonal group as structural group. By an *almost-complex structure* on E we mean a reduction of the structural group to the unitary group. By an *A-structure* on E I mean a fibre-preserving map $f: E \to E$ such that fx is orthogonal to x for all $x \in E$. For example, an almost-complex structure determines such a map through the action[2] of the scalar J such that $J^2 = -1$. Note that n must be even if an A-structure exists. When E is trivial this necessary condition is also sufficient.

I describe E as *homotopy-symmetric* if $1 \simeq u: E \to E$, by a fibre-preserving homotopy, where u denotes the antipodal map given by $ux = -x$. This condition also implies that n is even. An A-structure f on E determines a fibre-preserving homotopy f_t $(t \in I = [0, 1])$, where $f_t x = x \cos \pi t + f(x) \sin \pi t$, and so E is homotopy-symmetric. I assert that the converse holds in the stable range,[3] so that we have

THEOREM 1. *Let B be a finite complex such that* $\dim B \leqq n-4$. *Then E admits an A-structure if and only if E is homotopy-symmetric.*

A proof can be given as follows. Let $p: E \to B$ denote the fibration. Let E' denote the space of pairs (x, y), where $x, y \in E$, such that $px = py$ and such that x is orthogonal to y. We fibre E' over E with projection p' given by $p'(x, y) = x$. An A-structure f on E determines a cross-section $f': E \to E'$, where $f'x = (x, fx)$, and conversely a cross-section determines an A-structure. Let E'' denote the space of paths λ in E such that $p\lambda$ is stationary in B and such that $\lambda(0) = \lambda(1)$. We

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[2] We recall that the centre of the structural group acts on the bundle.

[3] The stable range, in relation to this problem, is not quite as extensive as the stable range of ordinary theory.

fibre E'' over E with projection p'' given by $p''\lambda=\lambda(0)$. Then $p''h=p'$, where $h: E'\to E''$ is the map defined by

$$h(x, y)(t) = x \cos \pi t + y \sin \pi t \qquad (t \in I).$$

A fibre-homotopy f_t of 1 into u determines a cross-section $f'': E\to E''$, where $f''(x)(t)=f_t(x)$, and conversely.

Consider the fibre S^{n-1} of the original fibration $p: E\to B$ which contains the basepoint $e\in E$. The fibre of $p': E'\to E$ over e can be identified with S^{n-2}, the equator orthogonal to e. The corresponding fibre of $p'': E''\to E$ can be identified with $\Omega(S^{n-1})$, the space of paths in S^{n-1} from e to $-e$, so that h maps $x\in S^{n-2}$ into the great semicircle through x. It follows that the Freudenthal suspension can be expressed as the composition

$$\pi_r(S^{n-2}) \underset{h_*}{\rightarrow} \pi_r(\Omega(S^{n-1})) \underset{\theta}{\rightarrow} \pi_{r+1}(S^{n-1}),$$

where θ denotes the Hurewicz isomorphism. Therefore h_* is injective for $r<2n-5$ and surjective for $r\leqq 2n-5$. However

$$\dim E = n - 1 + \dim B \leqq 2n - 5,$$

by hypothesis, and so the coefficient homomorphism

$$H^i(E, \pi_j(S^{n-2})) \rightarrow H^i(E, \pi_j(\Omega(S^{n-1}))),$$

induced by h_* is injective for $j<i$, surjective for $j\leqq i$. Hence it follows by obstruction theory, as in [2], that $p': E'\to E$ admits a cross-section if $p'': E''\to E$ does so. This proves Theorem 1.

Now consider the (reduced) Grothendieck group $\tilde{K}_R(B)$ formed from real vector bundles over B. If E is a sphere-bundle over B we denote by $[E]$ the class in $\tilde{K}_R(B)$ of the associated euclidean bundle. I say that an element $x\in\tilde{K}_R(B)$ is of A-type if $x=[E]$ where E is a sphere-bundle which admits A-structure. By an argument similar to that used to prove Theorem 1 we obtain

THEOREM 2. *Let B be a finite-dimensional complex. Let E be an $(n-1)$-sphere bundle over B where n is even and* $\dim B\leqq n-4$. *Then E admits A-structure if $[E]$ is of A-type.*

From this it is not difficult to show that the elements of A-type in $\tilde{K}_R(B)$ form a subgroup. Our main result is a determination of this subgroup as follows. Let

$$T: \quad K_R(B) \rightarrow K_R(SB)$$

denote the homomorphism given by taking the tensor product with the class of the canonical line bundle over the real projective line.

Let J have its standard meaning, as in [1], for example. Then we have

THEOREM 3. *An element* $x \in \tilde{K}_R(B)$ *is of A-type if and only if* $JT(x)=0$.

It is well known[4] that the kernel of T coincides with the image of the homomorphism

$$\tilde{K}_C(B) \to \tilde{K}_R(B)$$

given by taking the underlying real vector bundle. It follows that $T(x)=0$ if and only if $x=[E]$, where E is a sphere-bundle which admits almost-complex structure. It turns out, after a little calculation, that T and JT have the same kernel when B is a sphere or one of the ordinary projective spaces. However, the kernels are different when B is the Cayley projective plane, and so there exists a stable bundle over this 16-dimensional manifold which admits an A-structure but not an almost-complex structure.

Outside the stable range there are some fragmentary results. For example it is shown by G. Whitehead [5] that the tangent sphere-bundle to S^n admits an A-structure if and only if $n=2$ or 6. I do not know of any manifold where the tangent bundle admits an A-structure but not an almost-complex structure.

An A-structure f determined by an almost-complex structure has the properties that (i) $uf=fu$, and (ii) $f^2=u$, where u denotes the antipodal map as before. It might be interesting to study conditions for the existence of A-structures with one or both of these properties.

The following is an indication of the proof of Theorem 3. We represent points on the circle S by complex numbers of unit modulus. Let $p: E \to B$ be as before. Consider the space $\tilde{E}$ formed from $E \times S$ by identifying (x, z) with $(-x, -z)$ for all $x \in E$, $z \in S$. We fibre $\tilde{E}$ over $B \times S$ with projection $\tilde{p}$ given by $\tilde{p}(x, z)=(px, z^2)$. The sphere-bundle structure is completed in the obvious way. The vector bundle associated with $\tilde{E}$ is the tensor product of the vector bundle associated with E and the canonical line bundle over S.

Let S_+ denote the upper semicircle of S. Suppose that E is homotopy-symmetric. Then there exists a map $g: E \times S_+ \to E$ such that $p(x, z)=px$ for all $x \in E$, $z \in S_+$, and such that

$$g(x, 1) = x = g(-x, -1).$$

We extend g to a map $h: E \times S \to E$ so that $h(x, z)=h(-x, -z)$, for all $x \in E$, $z \in S$. Then $pk=l\tilde{p}$, where $k: \tilde{E} \to E$ is induced by h and

[4] I understand this version of a result of Bott's can be found in D. W. Anderson's unpublished thesis. An independent proof by R. Wood is also unpublished.

$l: B\times S\to B$ is the left projection. Since k maps one of the fibres of $\tilde{E}$ homeomorphically onto the corresponding fibre of E it follows that $\tilde{E}$ is fibre-homotopically equivalent to $l^*(E)$. In terms of K-theory this implies that $T([E])$ lies in the kernel of J. When E is a stable bundle the essential steps of this argument are reversible and so, after using Theorem 1, we arrive at Theorem 3 as asserted. Full details are given in [**3**].

These results can be extended to (right) vector bundles over F, where F denotes the field of real numbers, complex numbers or quaternions. Vector spaces over F are endowed with the usual inner product (see §20 of [**4**], for example) so that orthogonality is defined. Let E be a sphere-bundle over B with structural group the orthogonal group in the real case, the unitary group in the complex case, the symplectic group in the quaternionic case. By an *A-structure* on E I mean a fibre-preserving map $f: E\to E$ such that fx is orthogonal to x, in the appropriate sense, for all $x\in E$. Let D denote the unit ball consisting of elements $z\in F$ such that $|z|\leqq 1$, and let S denote the unit sphere, where $|z|=1$. I describe E as *homotopy-symmetric* if there exists a map $h: E\times D\to E$ such that

(i) $ph(x,z)=px \qquad (x\in E, z\in D)$,

(ii) $h(x,z)=x\cdot z \qquad (x\in E, z\in S)$.

By arguments similar to those used in the real case it can be shown that the existence of an A-structure implies homotopy-symmetry, and that the converse holds when B satisfies similar conditions to those in Theorem 1.

The notion of A-type is extended to $\tilde{K}_F(B)$, in the obvious way. There is no difficulty in generalizing Theorem 2. The generalization of Theorem 3 asserts that the elements of A-type coincide with the kernel of JT, where

$$T:\ \tilde{K}_F(B)\to \tilde{K}_R(B\wedge P_F(1))$$

is defined by taking the tensor product with the canonical left line bundle[5] over $P_F(1)$, the left projective line. Details are given in [**3**].

We describe an A-structure $f: E\to E$ as *equivariant* if

$$f(xz)=f(x)\cdot z \qquad (x\in E,\ z\in S).$$

Consider the case when B is a point-space and $E=S_n$, the unit sphere in the n-dimensional vector space over F. An A-structure f on S_n determines a nonsingular vector field, by taking the tangent at x to the great circle through fx. If f is equivariant we obtain an induced

[5] As defined in §4 of [**1**].

nonsingular vector field on the left $(n-1)$-dimensional projective space S_n/S. Since complex and quaternionic projective spaces have nonzero Euler number it follows that equivariant A-structures cannot exist unless F is real.

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