SOME NONZERO HOMOTOPY GROUPS OF SPHERES

BY EDWARD B. CURTIS

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1. The purpose of this note is to establish some nonzero elements in the homotopy groups of spheres. This results from unstabilizing a method of Adams. Namely, an Adams spectral sequence is used to detect elements in $\pi_{n+i}(S^n)$ for various n and i; in addition to the d and e invariants of Adams, the Hopf invariants are used to show that certain of these elements are nonzero. One consequence will be the following.

Consequence. The groups $\pi_{4+i}(S^4)$ are nonzero for all $i \geqq 0$.

2. Recall the mod-p-restricted lower central series spectral sequence (abbr: mod-p-RLCSSS), constructed as in [4], [5] and [10]. For each simplicial set X, form GX as in [6], filter GX by its mod-p-RLCS, and pass to the homotopy exact couple. The resulting spectral sequence we will label $E^r_{s,d}(X)$, where s = filtration and d = dimension. The results of [4, §(2.4)] show that for the sphere spectrum S, the term $E^1(S)$ of the mod-2-RLCSS is a ring Λ, with multiplicative generators λ_i for each $i \geqq 0$. An additive basis for $E^1(S)$ consists of all monomials $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_k}$, where $I = (i_1, \cdots, i_k)$ is a sequence of nonnegative integers with $2i_j \geqq i_{j+1}$ for $j = 1, 2, \cdots, k-1$. Call such monomials allowable. In the unstable case, the results of [4, §(5.4)] show that for the n-sphere S^n, $E^1(S^n)$ is the subvector space of Λ with basis all λ_I which are allowable and for which $i_1 < n$. Such a monomial $\lambda_I \in E^1(S^n)$, where $I = (i_1, \cdots, i_k)$, has filtration k, and dimension $n + \sum i_j$.

3. There is a short exact sequence of differential vector spaces:

$$0 \to E^1_{s,n+i}(S^n) \xrightarrow{i} E^1_{s,n+i+1}(S^{n+1}) \xrightarrow{h} E^1_{s-1,n+i+1}S^{2n+1}) \to 0$$

where i is the inclusion and h is defined on the allowable basis by

$$\begin{aligned} h(\lambda_j\lambda_I) &= \lambda_I \quad \text{for } j = n, \\ &= 0 \quad \text{for } j < n. \end{aligned}$$

From this, there derives a long exact sequence

$$(3.1) \qquad \cdots \to E^2(S^n) \xrightarrow{i_*} E^2(S^{n+1}) \xrightarrow{h_*} E^2(S^{2n+1}) \xrightarrow{\partial} \cdots.$$

It can be shown that h_* commutes with all differentials, and is induced by the Hopf-invariant in the SHP-sequence of Whitehead, James:

$$\cdots \to \pi_{n+1}(X^n) \xrightarrow{S} \pi_{n+i+1}(S^{n+1}) \xrightarrow{H} \pi_{n+i+1}(S^{2n+1}) \xrightarrow{P} \cdots .$$

From the sequence (3.1), some calculations in $E^2(S^n)$ can easily be made.

4. For each $m \geqq 0$, define functions $\phi_2(m), \phi_3(m), \phi_4(m), \phi_5(m), \phi(m)$ by the rules:

$m = 8k+$	0	1	2	3	4	5	6	7
$\phi_2(m) = 4k+$	0	1	2	3	4	4	5	4
$\phi_3(m) = 4k+$	0	1	2	3	3	4	3	4
$\phi_4(m) = 4k+$	0	1	2	3	3	4	4	4
$\phi_5(m) = 4k+$	0	1	2	3	3	4	4	4
$\phi(m) = 4k+$	0	1	2	3	3	3	3	4

The function $\phi(m)$ describes the Adams vanishing line: $\mathrm{Ext}_{A_2}^{s,t}(Z_2, Z_2) = 0$ for $s > \phi(t-s)$. Unstably, the functions $\phi_n(m)$ (set $\phi_n(m) = \phi(m)$ for $n \geqq 6$) also describe a vanishing line, possibly modulo a tower, as follows.

THEOREM. *$E^2_{s,n+i}(S^n) = 0$ for $s > \phi_n(i)$, except for the tower at $i = 0$, and the tower which occurs when n is even and $i = n-1$.*

This can be proven using the stable vanishing line $\phi(m)$ of Adams [1], (3.1), and downward induction.

COROLLARY. *In the 2-component of $\pi_{n+i}(S^n)$, each element has order $\leqq 2^{\phi_n(i)}$.*

This is of course the unstable analogue of [1, p. 69]. There is also a similar vanishing line for each prime p, and all together give a bound for the order of any element (of finite order).

5. Let P be the periodicity operator defined by the Massey product $P(x) = \{x, \lambda_0^4, \lambda_7\}$. The following table describes some (not all) nonzero elements in $E^2(S^n)$ near the vanishing line. They are cycles in every $E^r(S^n)$ for which they are defined, as the differentials on them land in the vanishing-zone or in a tower.

TABLE

Stem dim i	Filtration s	Minimum value of n	Element in $E^2(S^n)$	Stable element in Ext (Z_2, Z_2)
$8k$	$4k-1$	3	$P^{k-1}(\lambda_2\lambda_3^2)$	$P^{k-1}(c_0)$
$8k+1$	$4k$	2	$P^{k-1}(\lambda_1\lambda_2\lambda_3^2)$	$P^{k-1}(h_1c_0)$
	$4k+1$	3	$P^k(\lambda_1)$	$P^k(h_1)$
$8k+2$	$4k+2$	2	$P^k(\lambda_1^2)$	$P^k(h_1^2)$
$8k+3$	$4k+1$	5	$P^k(\lambda_3)$	$P^k(h_2)$
	$4k+2$	3	$P^k(\lambda_2\lambda_1)$	$P^k(h_0h_2)$
	$4k+3$	2	$P^k(\lambda_1^3)$	$P^k(h_0^2h_2)$
$8k+4$	$4k+2$	4	$P^k(\lambda_3\lambda_1)$	0
	$4k+3$	3	$P^k(\lambda_2\lambda_1^2)$	0
$8k+5$	$4k+3$	4	$P^k(\lambda_3\lambda_1^2)$	0
	$4k+4$	3	$P^k(\lambda_2\lambda_1^3)$	0
$8k+6$	$4k+4$	4	$P^k(\lambda_3\lambda_1^3)$	0
$8k+7$	$4k+4$	5	$P^k(\lambda_7\lambda_0^3)$	$P^k(h_0^3h_3)$

The elements $P^{k-1}(c_0)$, $P^{k-1}(h_1c_0)$, $P^k(h_7)$, $P^k(h_1^2)$, $P^k(h_2)$, $P^k(h_0h_2)$, $P^k(h_0^2h_2)$, $P^k(h_0^3h_3)$ are shown never to be boundaries in the stable Adams spectral sequence because of nonzero d or e invariants; see [2], [7], [8], [9]. Hence, by naturality of suspension, their precursors are never boundaries in each $E^r(S^n)$ of the mod-2-RLCSSS.

The Hopf-invariant $h_*: E^r(S^3)\to E^r(S^5)$ shows that the elements $P^k(\lambda_2\lambda_1^2)$, $P^k(\lambda_2\lambda_1^3)$ are not boundaries in any $E^r(S^3)$, since h_* of them are not boundaries in $E^r(S^5)$. Similarly, the elements $P^k(\lambda_3\lambda_1)$, $P^k(\lambda_3\lambda_1^2)$ and $P^k(\lambda_3\lambda_1^3)$ are never boundaries in any $E^r(S^4)$.

6. For odd primes p, the E^1-term of the mod-p-RLCSSS for odd spheres is described in [4, §8]. The analogous vanishing statement is $E^2_{s,n+i}(S^n)=0$, for all odd n, and $s>[i+3/2p-2]$. Also, in filtration k and dimension $3+2k(p-1)-1$, $E^2(S^3)$ has a single generator say a_k. As all differentials on a_k land in the vanishing zone, a_k is a permanent cycle; also, a_k is never a boundary, shown by a mod-p version of [9]. Thus a_k detects a nonzero class of order p in $\pi_{3+2k(p-1)-1}(S^3)$. Of course the element detected by a_k is just (a nonzero multiple of) Toda's α_k shown to be nonzero by Adams' e-invariant argument.

7. It is now easy to exhibit some nonzero homotopy classes, as each of the elements in the table detects a nonzero class in $\pi_*(S^n)$ for the corresponding value of n. Using also the elements $\alpha_k(3)$ for stems

$\equiv 7 \pmod 8$, there follows consequence (1). Further, $\pi_{3+i}(S^3)$ is nonzero at least for all $i \not\equiv 6 \pmod 8$, and hence also $\pi_{2+i}(S^2)$ is nonzero at least for all $i \not\equiv 7 \pmod 8$.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASSACHUSETTS 02139