IMMERSIONS AND SURGERIES OF TOPOLOGICAL MANIFOLDS

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In this announcement, we outline a version of Haefliger and Poenaru's Immersion Theorem [2] for topological manifolds. We then use our theorem to do surgery on topological manifolds, and obtain results such as the following: Let M^n be a closed, almost parallelizable topological manifold (that is, the tangent bundle of $M-p$ is trivial, $p\in M$) which has the homotopy type of a finite complex. Then by a sequence of surgeries, M can be reduced to an $[n/2-1]$ connected almost parallelizable manifold.

In order to state the Immersion Theorem we give the following definitions: Let M, M' and Q be topological manifolds, M a compact locally flat submanifold of the open manifold M', with $\dim M' = \dim Q$.

Write $\mathrm{Im}_{M'}(M, Q)$ for the semisimplicial complex of M' immersions of M in Q; a simplex of $\mathrm{Im}_{M'}(M, Q)$ is an immersion $f: \Delta\times U \to \Delta\times Q$ commuting with the projections on the standard simplex Δ, where U is a neighborhood of M in M'. Two such are identified if they agree on $\Delta\times$(a neighborhood of M in M').

Write $R(TM'|M, TQ)$ for the semisimplicial complex of representation germs of the tangent bundle of M' restricted to M in the tangent bundle of Q; a simplex of $R(TM'|M, TQ)$ is a microbundle map Φ of $\Delta\times TU$ in $\Delta\times TQ$ which commutes with projections on Δ, U a neighborhood of M in M', such that the map of $\Delta\times TU$ in $\Delta\times U\times Q$ given by $(t, u, u')\to(t, u, \pi\Phi(t, u, u'))$ is an immersion on a neighborhood of $\Delta\times$(the diagonal of M). Two such representations define the same representation germ if they agree on a neighborhood of $\Delta\times$(the diagonal of M.

Observe that if f is a simplex of $\mathrm{Im}_{M'}(M, Q)$, the map df defined as follows, is a simplex of $R(TM'|M, TQ)$: $df(t, u, u') = (t, f_t u, f_t u')$ where $u, u'\in U$, $f(t, u) = (t, f_t u)$. We now state the Immersion Theorem. Suppose M has a handlebody decomposition with all handles of index $<\dim Q$. Then the map $d: \mathrm{Im}_{M'}(M, Q)\to R(TM'|M, TQ)$ is a homotopy equivalence. R. Lashof has shown [6] that the hypothesis

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that M has a handlebody decomposition can be removed, in case $\dim M < \dim M'$.

We first prove the following

NEIGHBORHOOD n-ISOTOPY EXTENSION THEOREM.[2] *Let E^q be euclidean q-space, X a closed subset of E^q, $U \subset V$ neighborhoods of X in E^q. Let F: $U \times I^n \rightarrow V^q \times I^n$ be an n-isotopy, that is, an embedding which commutes with projections on I^n. Then there is an ambient n-isotopy H: $V^q \times I^n \rightarrow V^q \times I^n$ fixed outside a compact set with $H(F_0x, t) = F(x, t)$ for $x \in U'$, an open neighborhood of X contained in U.*

PROOF. For clarity, we give the argument for $n=1$. Applying the lemma below starting at any level $s \in I$ in either direction we obtain h_s and U_s so that (replacing F_0 by F_s) the theorem holds for $x \in U_s$ and t in an interval about s. We then use the compactness of the unit interval to construct the required isotopy H. Using an induction, this argument is generalized to prove the Neighborhood n-Isotopy Extension Theorem.

LEMMA. *There is an $\epsilon > 0$ and a level preserving homeomorphism h_0: $V \times [0, \epsilon] \rightarrow V \times [0, \epsilon]$ fixed outside a compact set with $h_0(F_0x, t) = F(x, t)$ for $x \in U_0$, $0 \leq t \leq \epsilon$, U_0 an open neighborhood of X contained in U.*

PROOF OF THE LEMMA. Using the arguments in [4] we obtain the following

Sublemma. Let h: $\Delta^i \times E^{q-i} \rightarrow \Delta^i \times E^{q-i}$ be a homeomorphism which is the identity on a neighborhood of $\partial\Delta^i \times E^{q-i}$. Then if h is close to the identity, there is a homeomorphism g: $\Delta^i \times E^{q-i} \rightarrow \Delta^i \times E^{q-i}$ which is the identity on a neighborhood of $\partial\Delta^i \times E^{q-i}$ and outside a compact set such that $g = h$ on a neighborhood of $\Delta^i \times 0$. Further, the map $h \rightarrow g$ is continuous in the compact-open topology.

Now let K be a finite complex, K^i the i-skeleton of K, with $U \supset K \supset X$. We will construct a sequence of level preserving homeomorphisms k^i such that $k^iF(x, t) = (F_0x, t)$ for x in a neighborhood of K^i, $t \in [0, \epsilon]$.

Assume k^{i-1} has been defined and let Δ be an i-simplex of K^i, $i \geq 0$. We may assume F_0 is the identity, so $k^{i-1}F$ is the identity on a neighborhood of $\partial\Delta$. Let $\Delta \times E^{q-i}$ be contained in a neighborhood of Δ; from the arguments in [5] we obtain an isotopy: $\Delta \times E^{q-i} \times [0, \epsilon] \rightarrow \Delta \times E^{q-i} \times [0, \epsilon]$ which agrees with $k^{i-1}F$ on a neighborhood of $[\Delta \times 0 \cup \partial\Delta \times E^{q-i}] \times [0, \epsilon]$.

[2] Robert Edwards has proved this result independently.

The Sublemma then provides homeomorphisms g_t^i with $g_t^i = k_t^{i-1}F_t$ on a neighborhood of $\Delta\times 0\cup\partial\Delta\times E^{q-i}$ and fixed outside $\Delta\times E^{q-i}$. Putting together the isotopies so obtained from each of the i-simplexes of K^i in turn we obtain the inverse of the required isotopy k^i. This completes the inductive step and the proof of the lemma.

We now prove a covering homotopy property for spaces of immersions. Let $A'\subset A$ be compact subsets of E^q. To simplify notation in what follows, we write $\mathrm{Im}(A, Q)$ and $R(TA, TQ)$ for $\mathrm{Im}_{E^q}(A, Q)$ and $R(TE^q|A, TQ)$ respectively. Note that if $F:\Delta\times U\to\Delta\times Q$ or $F:\Delta\times TU\to\Delta\times TQ$ is a simplex of $\mathrm{Im}(A, Q)$ or $R(TA, TQ)$, U a neighborhood of A, then the restriction ρF of F to $\Delta\times U'$ or $\Delta\times TU'$ is a simplex of $\mathrm{Im}(A', Q)$ or $R(TA', TQ)$, U' a neighborhood of A'.

THEOREM 1. *The restriction map* $\mathrm{Im}(A, Q)\to\mathrm{Im}(A', Q)$ *is a fibration, if* $A = D^k\times D^{n-k}$, $A' = \partial D^k\times D^{n-k}$, $k<\dim Q$.

This means the following: Let $F: I\times I^n\times U'\to I\times I^n\times Q$ and $F_0': I^n\times U\to I^n\times Q$ be level preserving immersions, with $F(0, t, u) = F_0'(t, u)$. Then there is a level preserving immersion $F': I\times I^n\times U\to I\times I_x^n Q$ with $F'(0, t, u) = F_0'(t, u)$ such that $F' = F$ on $I\times I^n\times U''$, U'' a neighborhood of A' contained in U'.

PROOF OF THEOREM 1. Let U_0' be a neighborhood of A' in E^q whose closure is compact and contained in U'. Since F is an immersion we can find $\epsilon>0$ and a level preserving embedding p: $[0, \epsilon]\times I^n\times U_0'\to[0, \epsilon]\times I^n\times U'$ so that $F_0 p_t = F_t$ for $0\leqq t\leqq\epsilon$, where we write $F(t, t', u) = (t, F_t(t', u))$.

Applying this result in either direction at any level t we obtain U_t', $I_t = [t-\epsilon(t), t+\epsilon(t)]$ and p^t such that $U_t'\subset U'$, $p^t: I_t\times I^n\times U_t'\to I_t\times I_x^n U'$ is an isotopy and $F_t p_s^t = F_s$, $s\in I_t$.

By the n-Isotopy Theorem there is a neighborhood U_i'' of A' contained in U_{i-1}' and an isotopy $H^t: I_t\times I^n\times U_{i-1}'\to I_t\times I^n\times U_{i-1}'$ fixed outside a compact set with $H_s^t p_t^t(t', u) = p_t^t(t', u)$, $u\in U_i''$, U_{i-1}' prescribed. As in [2], if $k<\dim Q$ we may reduce to the case in which this compact set lies inside $p_{s_{i-1}}^{t_i}(I^n\times U_{i-1}')$.

Since I is compact we can write $0 = t_0 < s_0 < t_1 < s_1 < \cdots < s_{k-1} < t_k = 1$ so that $[s_{i-1}, s_i]\subset(t_i-\epsilon(t_i), t_i+\epsilon(t_i))$ for all i. Suppose inductively that a level preserving immersion F': $[0, s_{i-1}]\times I^n\times U\to[0, s_{i-1}]\times I^n\times Q$ has been defined with $F_t'(t', u) = F_t(t', u)$ for $u\in U_{i-1}'$, a neighborhood of A' in U'.

Extend F' over $[0, s_i]\times I^n\times U$ as follows.

$$F_t'(t', u) = F_{t_i}H_t^{t_i}(H_{s_{i-1}}^{t_i})^{-1}P_{s_{i-1}}^{t_i}(t', u), \qquad u\in U_{i-1}',$$

$$F_t'(t', u) = F_{s_{i-1}}'(t', u), \qquad u\in U - U_{i-1}',\ s_{i-1}\leqq t\leqq s_i.$$

Note that for u in a neighborhood of $\partial U'$ in $\overline{U}'$, $H_t^{t_i}=1$ so that this extension is a well-defined immersion. A calculation shows that if $u\in U_i' = U_{i-1}'\cap U_{u_i}''$, $F_i'(t', u)=F_i(t', u)$. This completes the inductive step; Theorem 1 is proved by starting the induction with F_0'.

Let $D^k\times D^{n-k}\subset E^q$; we identify $\partial D^k\times D^{n-k+1}$ with a neighborhood of $\partial D^k\times D^{n-k}$ in $D^k\times D^{n-k}$. Let U, U' be neighborhoods of $D^k\times D^{n-k}$ and $\partial D^k\times D^{n-k+1}$ respectively in E^q, and let ϕ be the U' germ of an immersion f of U in Q.

Write $\mathrm{Im}_\phi(D^k\times D^{n-k}, Q)$ for the semisimplicial complex of E^q immersions of $D^k\times D^{n-k}$ in Q whose $\partial D^k\times D^{n-k+1}$ germ is equal to ϕ. Similarly, let $R_\phi(T(D^k\times D^{n-k}), TQ)$ be the semisimplicial complex of representation germs of TU in TQ whose restriction to U' is $d\phi$.

Now by an argument formally identical to that in [2], Theorem 1 implies

LEMMA 1. *The map* $d\colon \mathrm{Im}_\phi(D^k\times D^{n-k}, Q)\to R_\phi(T(D^k\times D^{n-k}), TQ)$ *is a homotopy equivalence if* $k<\dim Q$.

LEMMA 2. *The map* $d\colon \mathrm{Im}(\partial D^{k+1}\times D^{n-k}, Q)\to R(T(\partial D^{k+1}\times D^{n-k}), TQ)$ *is a homotopy equivalence if* $k<\dim Q$.

Now let M, M' and Q be topological manifolds, M a compact locally flat submanifold of the open manifold M', with $\dim M' =\dim Q$. Suppose M has a handlebody decomposition with all handles of index $<\dim Q$.

THE IMMERSION THEOREM. *The map* $d\colon \mathrm{Im}_{M'}(M, Q)\to R(TM'\,|\,M, TQ)$ *is a homotopy equivalence.*

PROOF. We argue by induction on the number of handles of M. Suppose $M=M_0\cup D^k\times D^{n-k}$, $M_0\cap D^k\times D^{n-k}=\partial D^k\times D^{n-k+1}$. Since M is locally flat in M', $D^k\times D^{n-k}$ is contained in a coordinate neighborhood in M'; thus by Theorem 1, the map $\mathrm{Im}_{M'}(M, Q)\to \mathrm{Im}_{M'}(M_0, Q)$ induced by restriction, is a fibration. The proof now proceeds by an argument formally identical to that in [2].

We now use the Immersion Theorem to do surgery on topological manifolds (see [5]). Let M^n be a topological manifold, TM the tangent microbundle of M and $f_0\colon S^p\to M$ a continuous map.

LEMMA 3. *If* $2p<n$ *and the induced bundle* f_0^*TM *is trivial, there is an embedding* $f\colon S^p\times D^{n-p}\to M$ *which represents the homotopy class of* f_0.

PROOF. Let $\pi\colon S^p\times R^{n-p}\to S^p$ be the natural projection. Then $(f_0\pi)^*TM$ is trivial, thus the standard trivialization of $T(S^p\times R^{n-p})$ induces a representation of $TR^n\,|\,S^p$ in $TM(S^p\times R^{n-p}\subset R^n)$. By the

Immersion Theorem, there is a regular homotopy class of E^n immersions of S^p in M corresponding to this representation. The result below shows that such a regular homotopy class contains an embedding, if $2p<n$. The proof of Lemma 3 is completed by noting that an E^n embedding of S^p in M restricts to an embedding of $S^p\times D^{n-p}$ in M.

Using Černavskiĭ's Theorem [1] and General Position arguments we can show the following

THEOREM. *Let M^n be a topological manifold and K a p-complex in E^n, with $2p<n$. Then any regular homotopy class of E^n immersions of K in M contains an immersion $f:U\to M$ with $f|K$ an embedding.*

Note that f is then an embedding on a neighborhood of K.

THEOREM 2. *Let M^n be an almost parallelizable topological manifold. Let $\lambda\in\pi_pM$, with $2p<n$. Then λ can be represented by an embedding $f: S^p\times D^{n-p}\to M$ such that the manifold $\chi(M, f)$ obtained from M by surgery is also almost parallelizable.*

PROOF OF THEOREM 2. Let Φ be a trivialization of $T(M-x)$. As in Lemma 3, we can find an embedding $f: S^p\times R^{n-p}\to M-x$ such that the trivialization of the restriction of TM to $f(S^p\times R^{n-p})$ induced by df from the standard trivialization of $T(S^p\times R^{n-p})$ is homotopic to $\Phi|f(S^p\times R^{n-p})$.

Now $\chi(M, f)=M-f(S^p\times \mathring{D}^{n-p})\cup D^{p+1}\times S^{n-p-1}$ where $(\theta, \theta')\in S^p\times S^{n-p-1}$ is identified with $f(\theta, \theta')$. Let

$$\chi' = S^p\times (R^{n-p}-\mathring{D}^{n-p})\cup D^{p+1}\times 1\times \mathring{D}^{n-p-1},$$

where $1\times\mathring{D}^{n-p-1}\subset\partial(D^1\times D^{n-p-1})=S^{n-p-1}$. Now the standard trivialization of $T(S^p\times R^{n-p})$ restricted to $S^p\times(R^{n-p}-\mathring{D}^{n-p})$ extends to a trivialization of $T\chi'$, identifying $(S^p\times 1)\times\mathring{D}^{n-p-1}\subset S^p\times D^1\times D^{n-p-1}\subset R^{p+1}\times R^{n-p-1}$ with $\partial D^{p+1}\times(1\times\mathring{D}^{n-p-1})$.

By the Covering Homotopy Theorem [8], this implies $\Phi|M-f(S^p\times\mathring{D}^{n-p})$ extends to a trivialization of $T(M-f(S^p\times\mathring{D}^{n-p})\cup\chi')$, where $(\theta, t\theta')\in S^p\times(R^{n-p}-\mathring{D}^{n-p})\subset\chi'$ is identified with $f(\theta,t\theta')$, $t\geq 1$. However, $M-f(S^p\times\mathring{D}^{n-p})\cup\chi'=\chi(M, f)-D^n\approx\chi(M, f)-y$, $y\in\mathring{D}^n$. Thus $T(\chi(M,f)-x\cup y)$ is trivial. We may suppose $x\cup y$ is contained in a coordinate neighborhood ψR^n, thus $T(\chi(M, f)-\psi(0))$ is trivial.

Using the surgery techniques of [3] we have applied Theorem 2 together with the Immersion Theorem to show the following

THEOREM 3. *Let M^{4k+1} be a closed almost parallelizable topological manifold, which has the homotopy type of a finite complex. Then M is triangulable as a piecewise linear manifold.*

R. Lashof has generalized this result in [6], thus we omit the proof.

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